Example 0.1. Consider a binary symmetric channel, with probability matrix

$$P = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$$

Find the capacity of the channel.

Solution:

We know, from the matrix $P$ that the probability of a correct transmission is

$$\Pr(Y = 0|X = 0) = P(Y = 1|X = 1) = 1 - p$$

and the probability of incorrect transmission for each symbol is given by

$$\Pr(Y = 0|X = 1) = \Pr(Y = 1|X = 0) = p.$$ 

We can choose the input distribution as: $\Pr(X = 0) = \alpha$ and $\Pr(X = 1) = 1 - \alpha$, $\alpha \in [0, 1]$.

We need $H(Y)$ and $H(Y|X)$ in order to derive $I(X;Y) = H(Y) - H(Y|X)$.

To compute $H(Y) = \sum_{y \in \{0,1\}} \Pr(Y = y) \log_2(\Pr(Y = y))$, we need $\Pr(Y = 0)$ and $\Pr(Y = 1)$.

$$\Pr(Y = 0) = \sum_{x \in \{0,1\}} \Pr(X = x) \cdot \Pr(Y = 0|X = x)$$

$$= \Pr(X = 0) \Pr(Y = 0|X = 0) + \Pr(X = 1) \Pr(Y = 0|X = 1)$$

$$= \alpha \cdot (1 - p) + (1 - \alpha) \cdot p = p + \alpha \cdot (1 - 2p) = D.$$ 

$$\Pr(Y = 1) = \sum_{x \in \{0,1\}} \Pr(X = x) \cdot P(Y = 1|X = x)$$

$$= \Pr(X = 0) \Pr(Y = 1|X = 0) + \Pr(X = 1) \Pr(Y = 1|X = 1)$$

$$= \alpha \cdot p + (1 - \alpha) \cdot (1 - p) = 1 - p - \alpha \cdot (1 - 2p) = 1 - D.$$ 

Therefore we have:

$$H(Y) = - \sum_{y \in \{0,1\}} \Pr(Y = y) \cdot \log_2(\Pr(Y = y))$$

$$= - [\Pr(Y = 0) \cdot \log_2(\Pr(Y = 0)) + \Pr(Y = 1) \cdot \log_2(\Pr(Y = 1))]$$

$$= - [D \cdot \log_2(D) + (1 - D) \cdot \log_2(1 - D)]$$

$$= H(D)$$ (the binary entropy function)

Next we compute the conditional entropy:

$$H(Y|X) = \sum_{x \in \{0,1\}} \Pr(X = x) H(Y|X = x)$$

$$= \Pr(X = 0) \cdot H(Y|X = 0) + \Pr(X = 1) \cdot H(Y|X = 1)$$

$$= - \Pr(X = 0) \cdot [\Pr(Y = 0|X = 0) \log_2(\Pr(Y = 0|X = 0)) + \Pr(Y = 1|X = 0) \log_2(\Pr(Y = 1|X = 0))]$$

$$- \Pr(X = 1) \cdot [\Pr(Y = 0|X = 1) \log_2(\Pr(Y = 0|X = 1)) + \Pr(Y = 1|X = 1) \log_2(\Pr(Y = 1|X = 1))]$$

$$= - \alpha \cdot [(1 - p) \log_2(1 - p) + p \log_2 p] - (1 - \alpha) \cdot [p \log_2(p) + (1 - p) \log_2(1 - p)]$$

$$= \alpha \cdot H(p) + H(p) - \alpha \cdot H(p) = H(p).$$
Thus we have:

\[
I(X;Y) = H(Y) - H(Y|X) = H(p + \alpha \cdot (1 - 2p)) - H(p)
\]

We have that \( C = \max_X I(X;Y) \). Note that \( H(Y|X) = H(p) \) does not depend on \( \alpha \), therefore to compute the maximum of \( I(X;Y) \), we only need to find the value of \( \alpha \) which maximizes \( H(p + (1 - 2p) \cdot \alpha) \). Further, we know that the binary entropy function \( H(x) \) takes its maximum value when \( x = 1/2 \) and that when \( \alpha = 1/2 \), the value \( p + \alpha \cdot (1 - 2p) = 1/2 \). Hence \( I(X;Y) \) reaches its maximum value when \( \alpha = 1/2 \). Then

\[
C = \max_\alpha (H(p + \alpha \cdot (1 - 2p)) - H(p)) = H(1/2) - H(p) = 1 - H(p)
\]

Note that when the crossover probability is \( p = 1/2 \), then \( C = 0 \), i.e. we have no information about the transmitted bit from the received bit.

**Example 0.2.** Given a Binary Erasure Channel with probability matrix

\[
P = \begin{pmatrix}
1-p & 0 \\
p & p \\
0 & 1-p
\end{pmatrix}
\]

Find the capacity of the channel.

**Solution:** We can represent the BEC as follows:

![Binary Erasure Channel Diagram]

We choose an input distribution \( \Pr(X = 0) = \alpha \) and \( \Pr(X = 1) = 1 - \alpha \).

As before we have to compute \( H(Y) - H(Y|X) \).

\[
H(Y|X) = \sum_{x\in\{0,1\}} \Pr(X = x)H(Y|X = x)
= \Pr(X = 0) \cdot H(Y|X = 0) + \Pr(X = 1) \cdot H(Y|X = 1)
\]

Now we have that

\[
H(Y|X = 0) = -\left[ \Pr(Y = 0|X = 0) \log_2(\Pr(Y = 0|X = 0)) + \Pr(Y = \epsilon|X = 0) \log_2(\Pr(Y = \epsilon|X = 0)) + \Pr(Y = 1|X = 0) \log_2(\Pr(Y = 1|X = 0)) \right]
= -(1 - p) \cdot \log_2(1 - p) + p \log_2 p = H(p)
\]
and in the same way $H(Y|X = 1) = H(p)$. So $H(Y|X) = \alpha H(p) + (1 - \alpha)H(p) = H(p)$.

Then $C = \max_\alpha (H(Y) - H(Y|X)) = \max_\alpha (H(Y) - H(p))$. We know that $H(Y) \leq \log_2 3$ (because in general $H(Y) \leq \log_2 m$ and in this case $m = 3$), but we cannot achieve this by any choice of the input distribution. So we have to work a bit harder, and compute $H(Y)$.

- $\Pr(Y = 0) = \Pr(X = 0) \Pr(Y = 0|X = 0) + \Pr(X = 1) \Pr(Y = 0|X = 1) = (1 - \alpha)(1 - p) + 0 = (1 - \alpha)(1 - p)$.
- $\Pr(Y = \epsilon) = \Pr(X = 0) \Pr(Y = \epsilon|X = 0) + \Pr(X = 1) \Pr(Y = \epsilon|X = 1) = p$
- $\Pr(Y = 1) = \Pr(X = 0) \Pr(Y = 1|X = 0) + \Pr(X = 1) \Pr(Y = 1|X = 1) = 0 + \alpha(1 - p) = (\alpha)(1 - p)$.

Then using these values to compute $H(Y)$, we get

$$H(Y) = H(p) + (1 - p)H(\alpha)$$

and

$$C = \max_\alpha H(p) + (1 - p)H(\alpha) - H(p) = \max_\alpha (1 - p)H(\alpha).$$

The max of $H(\alpha)$ is reached for $\alpha = 1/2$ and $H(1/2) = 1$, so $C = 1 - p$. 
