

A quantum analogue of Fourier analysis on the boolean cube

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QIPC Rome 2009

Talk based on joint work with Tobias Osborne

Fourier analysis

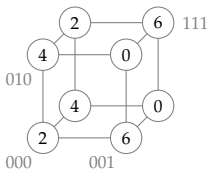
...traditionally looks like this:



- Given some function $f : \mathbb{R} \rightarrow \mathbb{R}$...
- ...we expand it in terms of **trigonometric** functions $\sin(kx)$, $\cos(kx)$...
- ...in an attempt to understand the **structure** of f .

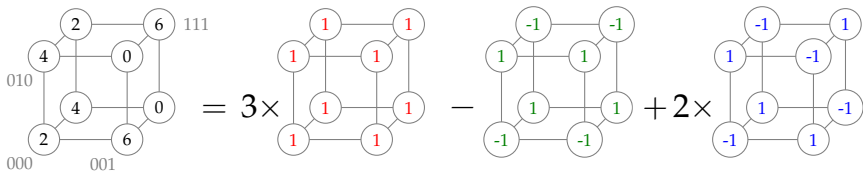
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- Given some function $f : \{0, 1\}^n \rightarrow \mathbb{R} \dots$
- ...we expand it in terms of **parity** functions...
- ...in an attempt to understand the **structure** of f .

This talk

- The classical theory of Fourier analysis on the boolean cube
- A quantum generalisation
- **Application:** Testing for Pauli operators
- The qubit depolarising channel
- **Application:** Spectra of k -local operators

Fourier analysis on the boolean cube

We expand functions $f : \{0, 1\}^n \rightarrow \mathbb{R}$ in terms of the parity functions

$$\chi_S(x) = (-1)^{\sum_{i \in S} x_i},$$

also known as the **characters** of \mathbb{Z}_2^n .

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Any $f : \{0, 1\}^n \rightarrow \mathbb{R}$ has the expansion

$$f = \sum_{S \subseteq \{1, \dots, n\}} \hat{f}_S \chi_S$$

for some $\{\hat{f}_S\}$ – the **Fourier coefficients** of f .

Applications of Fourier analysis on the boolean cube

This approach has led to new results in many areas of classical computer science, including:

- Probabilistically checkable proofs [Håstad '01; Dinur '07; ...]
- Decision tree complexity [Nisan & Szegedy '94]
- Influence of voters and fairness of elections [Kahn, Kalai, Linial '88; Kalai '02]
- Computational learning theory [Goldreich & Levin '89; Kushilevitz & Mansour '91; ...]
- **Property testing** [Bellare et al '95; Matulef et al '09; ...]

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- We want to output whether f has some property P , or is “far” from having property P , using a **constant** number of queries.
- Sample problem: Determine whether f is **linear**, or “far” from linear: i.e. differs from all linear functions in a constant fraction of places.

Applications in quantum computation

There have also been some recent applications of Fourier analysis to [quantum](#) computer science.

- Quantum algorithms for computational learning
[Bshouty & Jackson '95; Atici & Servedio '07]
- Quantum communication complexity
[Klauck '01; Gavinsky et al '07]
- Lower bounds on quantum locally decodable codes
[Ben-Aroya, Regev, de Wolf '08]
- Quantum algorithms with exponential speed-ups
[Roetteler '08; AM '08]

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- 1 Because we can: generalisations are generally interesting
- 2 The classical theory is very successful – maybe a quantum theory will be too
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Our generalisation: instead of decomposing functions $\{0, 1\}^n \rightarrow \mathbb{C}$, we decompose **linear operators** on the space of n qubits.

“Fourier analysis” for qubits

It turns out that a natural analogue of the characters of \mathbb{Z}_2 are the **Pauli matrices**:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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We write a tensor product of Paulis as

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Any n qubit linear operator f has an expansion

$$f = \sum_{\mathbf{s} \in \{0,1,2,3\}^n} \hat{f}_{\mathbf{s}} \chi_{\mathbf{s}}.$$

for some $\{\hat{f}_{\mathbf{s}}\}$ – the **Pauli coefficients** of f . This is our analogue of the Fourier expansion of a function $f : \{0, 1\}^n \rightarrow \mathbb{C}$.

Norms and closeness

Some definitions we'll need later:

- The (**normalised**) Schatten p -norm: for any d -dimensional operator M ,

$$\|M\|_p \equiv \left(\frac{1}{d} \sum_{j=1}^d \sigma_j^p \right)^{\frac{1}{p}},$$

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- **Closeness**: Let U and V be two linear operators. Then we say that f and g are **ϵ -close** if $\|U - V\|_2^2 \leq 4\epsilon$.

Quantum property testing

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We give a test (the [quantum Pauli test](#)) that has the following property.

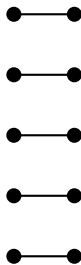
Proposition

Suppose that a unitary operator U passes the quantum Pauli test with probability $1 - \epsilon$. Then U is ϵ -close to a Pauli operator (with phase) $e^{i\phi}\chi_s$.

The test uses 2 queries (best known classical test uses 3).

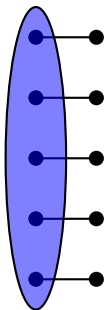
Quantum Pauli testing algorithm (sketch)

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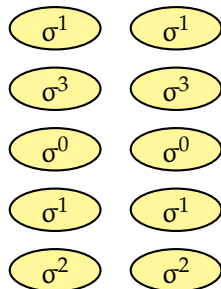
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σ^2

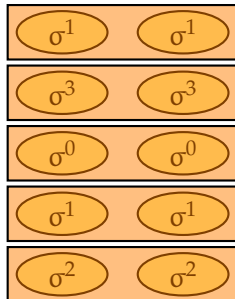
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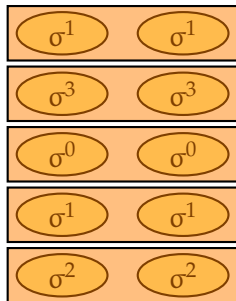
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- 5 Accept if all measurements say "yes".



It turns out that for the Pauli test $\Pr[\text{test accepts}] = \sum_s |\hat{U}_s|^4$, which implies the proposition by Parseval's equality.

The Pauli operators and depolarising noise

The Pauli expansion can help us understand the [qubit depolarising channel](#).

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- Let \mathcal{D}_ϵ be the qubit depolarising channel with noise rate $1 - \epsilon$, i.e.

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$$\mathcal{D}_\epsilon^{\otimes n}(\rho) = \sum_{\mathbf{s} \in \{0,1,2,3\}^n} \epsilon^{|\mathbf{s}|} \hat{\rho}_{\mathbf{s}} \chi_{\mathbf{s}}.$$

(this connection goes back at least a decade [[Bruss et al '99](#)], and was used in [[Kempe et al '08](#)] to give upper bounds on fault-tolerance thresholds)

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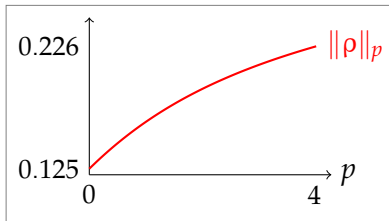
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We are interested in the [smoothing](#) effect of this channel.

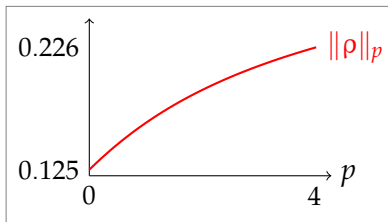
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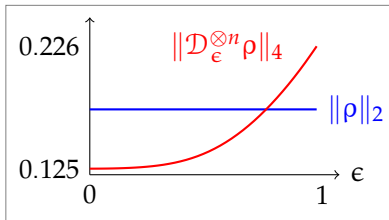


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Applying depolarising noise **smooths** ρ by reducing its higher norms:



Quantum hypercontractivity

Proposition

Let H be a Hermitian operator on n qubits and assume that $1 \leq p \leq 2 \leq q \leq \infty$. Then, provided that $\epsilon \leq \sqrt{\frac{p-1}{q-1}}$, we have

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- The quantum proof isn't a simple generalisation of the classical proof, but would be if the **maximum output $p \rightarrow q$ norm** were multiplicative!

Application: Spectra of k -local operators

A Hamiltonian H on n qubits is said to be k -local if it has a decomposition

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Our results show that the spectra of k -local operators are “smooth”. In particular:

- For any $q \geq 2$, $\|H\|_q \leq (q-1)^{k/2} \|H\|_2$
- $\text{rank}(H) \geq 2^{n-O(k)}$ (a quantum Schwartz-Zippel lemma)
- ...

Conclusions

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We still have many open conjectures... such as:

- **Conjecture:** There exists an efficient quantum property tester for dictators.
- **Conjecture:** Every traceless operator $U^2 = \mathbb{I}$ has an influential qubit: there is a j such that $\| \text{tr}_j U \otimes \mathbb{I}/2 - U \|_2^2 = \Omega((\log n)/n)$.
- ...

The end

Further reading:

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- “Learning and testing algorithms for the Clifford group”, Richard Low, [arXiv:0907.2833](https://arxiv.org/abs/0907.2833).
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Thanks for your time!

Application: A quantum FKN theorem

- The classical **FKN** (Friedgut-Kalai-Naor) theorem: Let f be a boolean function. Then, if $\sum_{|S|>1} \hat{f}_S^2 < \epsilon$, f is $O(\epsilon)$ -close to depending on 1 variable (being a **dictator**).

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Quantum FKN theorem

Let U be a unitary operator on n qubits with eigenvalues ± 1 . If

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then there is a constant K such that U is $K\epsilon$ -close to being a dictator (acting non-trivially on only 1 qubit) or the identity.

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We give a quantum algorithm that outputs the **large Pauli coefficients** of U . If U is almost completely determined by these, this is sufficient to approximately learn U .

Computational learning of unitary operators

“Quantum Goldreich-Levin” algorithm

Given oracle access to a unitary U , and given $\gamma, \delta > 0$, there is a poly $\left(n, \frac{1}{\gamma}\right) \log\left(\frac{1}{\delta}\right)$ -time algorithm which outputs a list $L = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_m\}$ such that with prob. $1 - \delta$: (1) if $|\hat{U}_{\mathbf{s}}| \geq \gamma$, then $\mathbf{s} \in L$; and (2) if $\mathbf{s} \in L$, $|\hat{U}_{\mathbf{s}}| \geq \gamma/2$.

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Example: learning dynamics of a 1D spin chain. Informally:

Theorem

Let H be a Hamiltonian corresponding to an n -site spin chain, and let $t = O(\log n)$. Then we can approximately learn the operators $\sigma_j^s(t) \equiv e^{-itH} \sigma_j^s e^{itH}$ with poly(n) uses of e^{itH} .

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So we can predict the outcome of measuring σ^s on site j after a short time, **on average** over all input states.