Exercises 20 Worked Answers

1. Find the matrices corresponding to the following transformations

   (a) Reflection in the y-axis – this is $y \rightarrow -y$, so the (prefix) matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

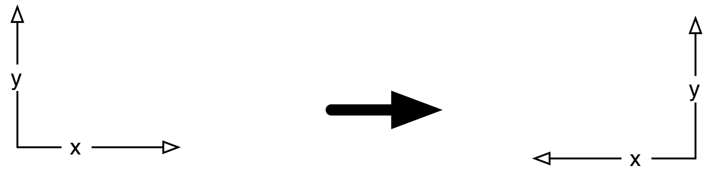

Figure 1: Reflection in y

   (b) 90° clockwise rotation corresponds to exchanging $-x$ and y:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

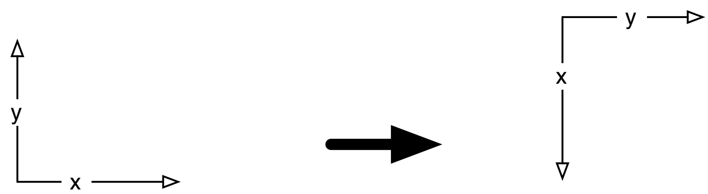

Figure 2: Clockwise quarter turn

   (c) A 180° rotation negates both x and y, so

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

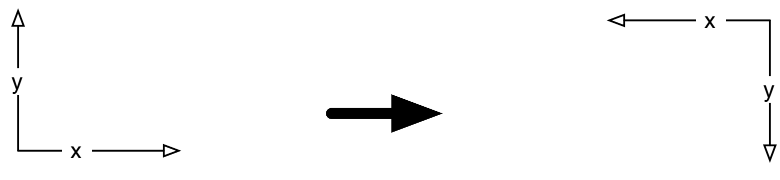

Figure 3: 180 degree rotation

   (d) A reflection in $y = -x$ swaps and negates x and y

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

   (e) From the figure, we see that $x \rightarrow 2x$ and $y \rightarrow y + x$ ($x = a$ here), so the matrix is

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

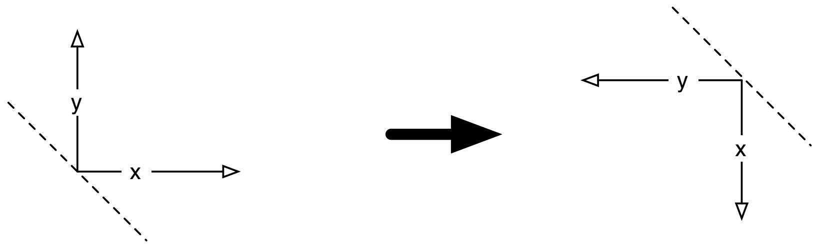

Figure 4: Reflection in $-x$

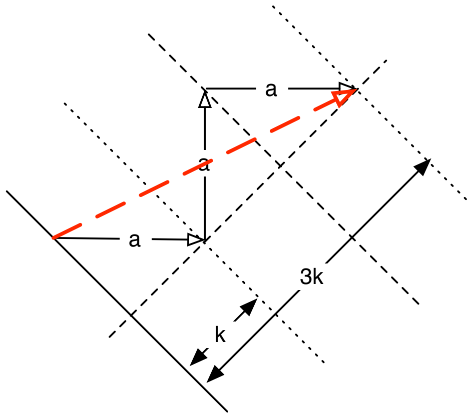

Figure 5: Scale by 3 perpendicular to $-x$

2. Find the matrix transforming $2, -3 \rightarrow 4, 14$ and $1, 3 \rightarrow 11, -2$. This gives us an equation like

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$$

and another one like

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \end{bmatrix}$$

So we multiply out the matrices and get the equations

$$2a - 3b = 4 \quad (1)$$

$$2c - 3d = 14 \quad (2)$$

$$a + 3b = 11 \quad (3)$$

$$c + 3d = -2 \quad (4)$$

Four equations in four unknowns, so we can use the normal process of reduction to solve them e.g. **2** - 2×**4** gives $-9d = 18, d = -2$ and so on.

3. (a) To work this out forwards we just do the multiplication so

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

   (b) To do it in reverse we require

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}^{-1}$$

   so we can do this using the general technique like:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/5 & 1/5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3/5 & -2/5 \\ 0 & 1 & 1/5 & 1/5 \end{bmatrix}$$

and then use the inverse:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/5 & -2/5 \\ 1/5 & 1/5 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 11 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

4. (a) After the multiplication we get

$$A_1 = (5, -3), B_1 = (15, -9), C_1 = (13, -7), D_1 = (3, -1)$$

(b) The determinant of the matrix is the area of the unit square after being transformed; Since the area of the original square is 4, this gives $4 \cdot |\det \begin{bmatrix} 5 & -1 \\ -3 & 1 \end{bmatrix}| = 8$

(c) Computing the inverse again:

$$\begin{bmatrix} 5 & -1 & 1 & 0 \\ -3 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1/5 & 1/5 & 0 \\ -3 & 1 & 0 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 3/2 & 5/2 \end{bmatrix}$$

So the inverse is

$$\begin{bmatrix} 1/2 & 1/2 \\ 3/2 & 5/2 \end{bmatrix}$$

5. (a) First we have $T = Q \times P$, so

$$T = \begin{bmatrix} -1 & 5 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 18 & 14 \\ 6 & 5 \end{bmatrix}$$

(b) Each unit of the area of the first shape is made into $|\det P|$ units of the second shape, and then each unit of area of the second shape becomes $|\det Q|$ of the third, so final area is $5 \cdot |\det P| \cdot |\det Q|$. However, we can also just say $5 \cdot |\det T|$, which gives $5 \cdot |18 \cdot 5 - 14 \cdot 6| = 30$.

6. There is a formal way of doing this (convert the matrix to reduced row-echelon form, and then the dimension of the space it projects into is given by the rank of the matrix), but I think in this case it's more instructive to observe what's really going on:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -2 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

This is saying that the result is a *linear combination of the columns:*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = x \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} + y \cdot \begin{bmatrix} 6 \\ -4 \end{bmatrix}$$

This interpretation of the equation immediately shows (a) why this is called linear algebra but (b) that the outcomes of the transformation must lie on a line. Since the second column is double the first column contributions from y point in the same direction as those from x, i.e. along the line $\mathbf{r} = \lambda \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This is a line because it's described by a single degree of freedom, the variable λ!, and if we were to rotate the axes through $\tan^{-1} 2/3$ anti-clockwise we'd see the matrix become something of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ which very obviously projects everything onto the x axis.