van Emde Boas trees

Benjamin Sach
Dictionaries

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In the **Cuckoo hashing** scheme:
- Every **lookup** and every **delete** takes *O(1)* worst-case time,
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These are very natural operations that the **Hashing**-based solutions that we have seen are very unsuited to
What could we use instead?

We could use a self-balancing binary search tree... like a 2-3-4 tree, a red-black tree or an AVL tree.
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All three of these data structures support:

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they are also deterministic.
van Emde Boas Trees

In this lecture, we will see the van Emde Boas (vEB) tree which stores a set $S$ of integer keys from a universe $U = \{1, 2, 3, 4 \ldots u\}$ (i.e. $u = |U|$).

Five operations will be supported:

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[Diagram showing predecessor and successor relationships]
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**Example:** If $U = \{1, 2, 3, 4 \ldots 100 \cdot n\}$, you get $O(\log \log n)$ time and $O(n)$ space.

![Diagram of the vEB tree showing the predecessor and successor operations for a key $k$.]
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and the space used is $O(u)$

and it is a deterministic data structure

**Example:** If $U = \{1, 2, 3, 4 \ldots n^3\}$, you get $O(\log \log n)$ time and $O(n^3)$ space
Attempt 1: a big array

Build an array of length $u$...
Attempt 1: a big array

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$
Attempt 1: a big array

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations add, delete and lookup all take $O(1)$ time.
Attempt 1: a big array

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations add, delete and lookup all take $O(1)$ time.
Attempt 1: a big array

Build an array of length \( u \)…

\[
A[i] = 1 \text{ iff } i \text{ is in } S
\]

\[\begin{array}{cccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\end{array}\]

The operations \textbf{add}, \textbf{delete} and \textbf{lookup} all take \( O(1) \) time.
**Attempt 1:** a big array

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations *add*, *delete* and *lookup* all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length $u$...

\[
A[i] = 1 \text{ iff } i \text{ is in } S
\]

The operations **add**, **delete** and **lookup** all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length $u$ ...

$A[i] = 1$ iff $i$ is in $S$

The operations *add*, *delete* and *lookup* all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations **add**, **delete** and **lookup** all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length $u$ . . .

$A[i] = 1$ iff $i$ is in $S$

The operations add, delete and lookup all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length $u$...

$$A[i] = 1 \text{ iff } i \text{ is in } S$$

The operations add, delete and lookup all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations *add*, *delete* and *lookup* all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length $u$...

\[ A[i] = 1 \text{ iff } i \text{ is in } S \]

The operations **add**, **delete** and **lookup** all take $O(1)$ time.
**Attempt 1: a big array**

Build an array of length $u$...

\[ A[i] = 1 \text{ iff } i \text{ is in } S \]

The operations add, delete and lookup all take $O(1)$ time.
Attempt 1: a big array

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations add, delete and lookup all take $O(1)$ time.

...looks good so far!
 Attempt 1: a big array

Build an array of length $u$…

$A[i] = 1$ iff $i$ is in $S$

The operations add, delete and lookup all take $O(1)$ time. …looks good so far!

What about the predecessor operation?
**Attempt 1: a big array**

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations **add**, **delete** and **lookup** all take $O(1)$ time.

...looks good so far!

What about the predecessor operation?

`predecessor(11)`
**Attempt 1: a big array**

Build an array of length $u$ ...

\[ A[i] = 1 \text{ iff } i \text{ is in } S \]

`predecessor(11)`

The operations `add`, `delete` and `lookup` all take $O(1)$ time.

...looks good so far!

What about the predecessor operation?
Attempt 1: a big array

Build an array of length $u$...

\[ A[i] = 1 \text{ iff } i \text{ is in } S \]

The operations \textit{add}, \textit{delete} and \textit{lookup} all take $O(1)$ time.

...looks good so far!

What about the \textit{predecessor} operation?
Attempt 1: a big array

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations add, delete and lookup all take $O(1)$ time.

\[\text{predecessor}(11)\]

...looks good so far!
Attempt 1: a big array

Build an array of length \( u \)...

\[
A[i] = 1 \text{ iff } i \text{ is in } S
\]

The operations \textit{add}, \textit{delete} and \textit{lookup} all take \( O(1) \) time.

...looks good so far!

\text{predecessor}(11)
**Attempt 1: a big array**

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations add, delete and lookup all take $O(1)$ time.

...looks good so far!
**Attempt 1: a big array**

Build an array of length $u$...

$A[i] = 1$ iff $i$ is in $S$

The operations add, delete and lookup all take $O(1)$ time.

...looks good so far!

The predecessor and successor operations take $O(u)$ time.
**Attempt 1: a big array**

Build an array of length $u$...

\[
A[i] = 1 \text{ iff } i \text{ is in } S
\]

The operations **add**, **delete** and **lookup** all take $O(1)$ time.

...looks good so far!

The **predecessor** and **successor** operations take $O(u)$ time.
**Attempt 1: a big array**

Build an array of length $u$…

$$A[i] = 1 \text{ iff } i \text{ is in } S$$

The operations add, delete and lookup all take $O(1)$ time. 
…looks good so far!

The predecessor and successor operations take $O(u)$ time  
…not so good!
Attempt 2: a constant height tree
(on top of a big array)
Attempt 2: a constant height tree
(on top of a big array)

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits
Attempt 2: a constant height tree

(on top of a big array)

A

\[
\begin{array}{cccccccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\hline
\sqrt{u} & \sqrt{u} & \sqrt{u} & \sqrt{u} & u \\
\end{array}
\]

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits
Attempt 2: a constant height tree

(on top of a big array)

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits
Attempt 2: a constant height tree
(on top of a big array)

\( C \) is called the summary of \( A \)

\[ \sqrt{u} \]

\[ C \]

\[ 1 \quad 1 \quad 0 \quad 1 \]

\[ \sqrt{u} \]

\[ \sqrt{u} \]

\[ u \]

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

$\sqrt{u}$

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.
Attempt 2: a constant height tree
(on top of a big array)

\( C \) is called the \textit{summary} of \( A \)

\[ \text{this is 1 if any bit in the child block is 1} \]

\[ \text{lookup}(12) \]

Split \( A \) into \( \sqrt{u} \) \textit{blocks} each containing \( \sqrt{u} \) bits

The \textit{lookup} and \textit{add} operations take \( O(1) \) time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

A constant height tree is called the **summary** of $A$.

- This is 1 if any bit in the child block is 1.

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits.

The lookup and add operations take $O(1)$ time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.
**Attempt 2: a constant height tree**  
*(on top of a big array)*

$C$ is called the *summary* of $A$

- This is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ *blocks* each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.
**Attempt 2:** a constant height tree  
*(on top of a big array)*

\( C \) is called the *summary* of \( A \)

- **this is 1 if** any bit in the child block is 1

\[ \text{add}(9) \]

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

The **lookup** and **add** operations take \( O(1) \) time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$. This is 1 if any bit in the child block is 1.

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits.

The lookup and add operations take $O(1)$ time.
Attempt 2: a constant height tree
(on top of a big array)

\( C \) is called the summary of \( A \)

This is 1 if any bit in the child block is 1

\[ \sqrt{u} \]

\[ \begin{array}{cccc}
  1 & 1 & 1 & 1 \\
\end{array} \]

\( C \)

\[ \begin{array}{ccccccccccccc}
  0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
\end{array} \]

\( \sqrt{u} \)

\( \sqrt{u} \)

\( \sqrt{u} \)

\( \sqrt{u} \)

\( u \)

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.
**Attempt 2: a constant height tree**

*(on top of a big array)*

$C$ is called the *summary* of $A$

This is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
Attempt 2: a constant height tree
(on top of a big array)

\( C \) is called the summary of \( A \)

this is 1 if any bit in the child block is 1

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
\end{array}
\]

\( A \)

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{array}
\]

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
Attempt 2: a constant height tree
(on top of a big array)

\( C \) is called the \textit{summary} of \( A \)

this is 1 if any bit in the child block is 1

delete(7)

\( \sqrt{u} \)

Split \( A \) into \( \sqrt{u} \) \textit{blocks} each containing \( \sqrt{u} \) bits

The \textit{lookup} and \textit{add} operations take \( O(1) \) time.

The operations \textit{delete}, \textit{predecessor} and \textit{successor} take \( O(\sqrt{u}) \) time.
**Attempt 2:** a constant height tree  
*(on top of a big array)*

$C$ is called the *summary* of $A$  

This is 1 if any bit in the child block is 1  

To determine this bit, you have to look through *this block*.

Split $A$ into $\sqrt{u}$ *blocks* each containing $\sqrt{u}$ bits.

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$.

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits.

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.

This is 1 if any bit in the child block is 1.

delete(7)
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the \textit{summary} of $A$.

This is 1 if any bit in the child block is 1.

Split $A$ into $\sqrt{u}$ \textit{blocks} each containing $\sqrt{u}$ bits.

The \textit{lookup} and \textit{add} operations take $O(1)$ time.

The operations \textit{delete}, \textit{predecessor} and \textit{successor} take $O(\sqrt{u})$ time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

\(C\) is called the *summary* of \(A\)

this is 1 if any bit in the child block is 1

\[
delete(9)
\]

\[
\begin{array}{c}
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 1 & \text{?} & 1 \\
\sqrt{u} & \sqrt{u} & \sqrt{u} & \sqrt{u} \\
\end{array}
\]

Split \(A\) into \(\sqrt{u}\) blocks each containing \(\sqrt{u}\) bits

The lookup and add operations take \(O(1)\) time.

The operations delete, predecessor and successor take \(O(\sqrt{u})\) time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

\[ C \] is called the *summary* of \( A \)

This is 1 if any bit in the child block is 1

To determine this bit you have to look through *this block*

Split \( A \) into \( \sqrt{u} \) *blocks* each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
**Attempt 2:** a constant height tree  
(on top of a big array)

\( C \) is called the **summary** of \( A \).  

This is 1 if any bit in the child block is 1.

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits.

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
 Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

predecessor(14)

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
**Attempt 2:** a constant height tree  
*(on top of a big array)*

\[ C \] is called the **summary** of \[ A \]

\[ \text{this is } 1 \text{ if any bit in the child block is } 1 \]

\[ \text{predecessor}(14) \]

Split \[ A \] into \( \sqrt{u} \) **blocks** each containing \( \sqrt{u} \) bits

The **lookup** and **add** operations take \( O(1) \) time.

The operations **delete**, **predecessor** and **successor** take \( O(\sqrt{u}) \) time.
Attempt 2: a constant height tree

(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

\(C\) is called the **summary** of \(A\)

this is 1 if any bit in the child block is 1

\[
\text{predecessor}(14)
\]

Split \(A\) into \(\sqrt{u}\) **blocks** each containing \(\sqrt{u}\) bits

The **lookup** and **add** operations take \(O(1)\) time.

The operations **delete**, **predecessor** and **successor** take \(O(\sqrt{u})\) time.
Attempt 2: a constant height tree
(on top of a big array)

\( C \) is called the summary of \( A \)

this is 1 if any bit in the child block is 1

\[ \sqrt{u} \]

predecessor(14)

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

\( C \) is called the *summary* of \( A \)

this is 1 if any bit in the child block is 1

\[
\begin{array}{cccccccccccccccc}
\text{1} & \text{1} & \text{1} & \text{1} & \text{1} & \text{0} & \text{0} & \text{0} & \text{0} & \text{0} & \text{0} & \text{0} & \text{1} & \text{0} & \text{0} & \text{0} \\
\text{1} & \text{2} & \text{3} & \text{4} & \text{5} & \text{6} & \text{7} & \text{8} & \text{9} & \text{10} & \text{11} & \text{12} & \text{13} & \text{14} & \text{15} & \text{16} \\
\end{array}
\]

Split \( A \) into \( \sqrt{u} \) *blocks* each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
**Attempt 2:** a constant height tree

*(on top of a big array)*

$C$ is called the **summary** of $A$

this is 1 if any bit in the child block is 1

**predecessor(14)**

Split $A$ into $\sqrt{u}$ **blocks** each containing $\sqrt{u}$ bits

The **lookup** and **add** operations take $O(1)$ time.

The operations **delete**, **predecessor** and **successor** take $O(\sqrt{u})$ time.
**Attempt 2**: a constant height tree

*(on top of a big array)*

\( C \) is called the \textit{summary} of \( A \)

\[
\text{this is } 1 \text{ if any bit in the child block is } 1
\]

\[
\text{predecessor}(14)
\]

Split \( A \) into \( \sqrt{u} \) \textit{blocks} each containing \( \sqrt{u} \) bits

The \textit{lookup} and \textit{add} operations take \( O(1) \) time.

The operations \textit{delete}, \textit{predecessor} and \textit{successor} take \( O(\sqrt{u}) \) time.
**Attempt 2:** a constant height tree  
(on top of a big array)

\( C \) is called the *summary* of \( A \)

this is 1 if any bit in the child block is 1

predecessor(14)

Split \( A \) into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
**Attempt 2:** a constant height tree  
*(on top of a big array)*

\(C\) is called the **summary** of \(A\)

This is 1 if any bit in the child block is 1

**C**

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
1 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 0 & 1 & 0 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Split \(A\) into \(\sqrt{u}\) **blocks** each containing \(\sqrt{u}\) bits

The **lookup** and **add** operations take \(O(1)\) time.

The operations **delete**, **predecessor** and **successor** take \(O(\sqrt{u})\) time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the summary of $A$

this is 1 if any bit in the child block is 1

predecessor(14)

Split $A$ into $\sqrt{u}$ blocks each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations delete, predecessor and successor take $O(\sqrt{u})$ time.
Attempt 2: a constant height tree
(on top of a big array)

$C$ is called the \textit{summary} of $A$

this is 1 if
any bit in the child block is 1

$C$ is called the \textit{summary} of $A$

In the worst case we look at
all of $C$ and all of two \textit{blocks}

Split $A$ into $\sqrt{u}$ \textit{blocks} each containing $\sqrt{u}$ bits

The lookup and add operations take $O(1)$ time.

The operations \textit{delete}, predecessor and successor take $O(\sqrt{u})$ time.
Attempt 2: a constant height tree
(on top of a big array)

A is called the summary of A

this is 1 if any bit in the child block is 1

In the worst case we look at all of C and all of two blocks (successor is the same)

Split A into \( \sqrt{u} \) blocks each containing \( \sqrt{u} \) bits

The lookup and add operations take \( O(1) \) time.

The operations delete, predecessor and successor take \( O(\sqrt{u}) \) time.
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

there is a whole lot more universe in here
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

there is a whole lot more universe in here
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

there is a whole lot more universe in here
Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

We can think of each block as a ‘little’ universe of size $\sqrt{u}$.

There is a whole lot more universe in here.

$\sqrt{u}$ $\sqrt{u}$ $\sqrt{u}$ $\sqrt{u}$

$\sqrt{u}$ $\sqrt{u}$ $\sqrt{u}$ $\sqrt{u}$

$u$
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

We can think of each block as a ‘little’ universe of size $\sqrt{u}$.
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

we can think of each block as a ‘little’ universe of size $\sqrt{u}$

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

We can think of each block as a ‘little’ universe of size $\sqrt{u}$.

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots, \sqrt{u}\}$.
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

we can think of each block as a ‘little’ universe of size $\sqrt{u}$

For block $i$, we build a data structure $B[i]$ which stores elements from \{1, 2, 3, \ldots, $\sqrt{u}$\}

$x$ is stored in $B[i]$ iff $(x + (i - 1)\sqrt{u}) \in S$
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

We can think of each block as a 'little' universe of size $\sqrt{u}$.

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$.

$x$ is stored in $B[i]$ iff $(x + (i - 1)\sqrt{u}) \in S$.

*(this is just to deal with the offset from the start of the real universe)*
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

For block $i$, we build a data structure $B[i]$

which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$

$x$ is stored in $B[i]$ iff $(x + (i - 1)\sqrt{u}) \in S$

(this is just to deal with the offset from the start of the real universe)
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

For block $i$, we build a data structure $B[i]$

which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$

$x$ is stored in $B[i]$ iff $(x + (i - 1)\sqrt{u}) \in S$
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots, \sqrt{u}\}$.

$x$ is stored in $B[i]$ iff $(x + (i - 1)\sqrt{u}) \in S$

We also build a summary data structure $C$ which stores elements from $\{1, 2, 3, \ldots, \sqrt{u}\}$.

$i$ is stored in $C$ iff $B[i]$ is non-empty.
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

For block $i$, we build a data structure $B[i]$ which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$

$x$ is stored in $B[i]$ iff $(x + (i - 1)\sqrt{u}) \in S$

We also build a summary data structure $C$ which stores elements from $\{1, 2, 3, \ldots \sqrt{u}\}$

$i$ is stored in $C$ iff $B[i]$ is non-empty
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

How should we build $B[1], B[2], \ldots B[\sqrt{u}]$ and $C$?
An abstract view

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Recursion!
An abstract view

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How should we build $B[1], B[2], \ldots B[\sqrt{u}]$ and $C$?

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Each $B[i]$ has universe $\{1, 2, 3, \ldots \sqrt{u}\}$
An abstract view

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How should we build $B[1], B[2], \ldots B[\sqrt{u}]$ and $C$?

Recursion!

Each $B[i]$ has universe $\{1, 2, 3, \ldots \sqrt{u}\}$

We recursively split this into $\frac{\sqrt{u}}{4}$ blocks each associated with $\frac{\sqrt{u}}{4}$ elements...
An abstract view

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

How should we build $B[1], B[2], \ldots B[\sqrt{u}]$ and $C$?

Recursion!

Each $B[i]$ has universe $\{1, 2, 3, \ldots \sqrt{u}\}$

We recursively split this into $\frac{1}{4}\sqrt{u}$ blocks each associated with $\frac{1}{4}\sqrt{u}$ elements...

eventually (after some more work), this will lead to an $O(\log \log n)$ time solution.
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

$\sqrt{u}$

$C$


$\sqrt{u}$  $\sqrt{u}$  $\sqrt{u}$  $\sqrt{u}$

$u$  $\sqrt{u}$  $\sqrt{u}$  $\sqrt{u}$

How do we perform the operations?
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in  
(this takes $O(1)$ time with a little bit twiddling)

**Step 2** If $B[i]$ is empty, add $i$ to $C$

**Step 3** add $x$ to $B[i]$ (suitably adjusting the offset from the start of $B[i]$)
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**Step 2** If $B[i]$ is empty, add $i$ to $C$

**Step 3** add $x$ to $B[i]$ (suitably adjusting the offset from the start of $B[i]$)

We actually insert $x'$ where $x = (x' + (i - 1)\sqrt{u})$.
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**Step 1** Determine which $B[i]$ the element $x$ belongs in  
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Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

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**Step 2** If $B[i]$ is empty, add $i$ to $C$

**Step 3** add $x$ to $B[i]$ (suitably adjusting the offset from the start of $B[i]$)
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

![Diagram showing the split of the universe into blocks with elements.](image)
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

To perform predecessor($x$):

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** Compute the predecessor of $x$ in $B[i]$

(suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:

Compute $j = \text{predecessor}(i)$ in $C$

Compute the predecessor of $x$ in $B[j]$

(suitably adjusting the offset from the start of $B[j]$)
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** Compute the predecessor of $x$ in $B[i]$  
(suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:  
Compute $j = \text{predecessor}(i)$ in $C$  
Compute the predecessor of $x$ in $B[j]$  
(suitably adjusting the offset from the start of $B[j]$)
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** Compute the predecessor of $x$ in $B[i]$

(suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:  
Compute $j = \text{predecessor}(i)$ in $C$
Compute the predecessor of $x$ in $B[j]$

(suitably adjusting the offset from the start of $B[j]$)
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

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**Step 2** Compute the predecessor of $x$ in $B[i]$ (suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:
- Compute $j = \text{predecessor}(i)$ in $C$
- Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$)
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

To perform `predecessor(x)`:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** Compute the predecessor of $x$ in $B[i]$  
(suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:
  Compute $j = \text{predecessor}(i)$ in $C$
  Compute the predecessor of $x$ in $B[j]$  
(suitably adjusting the offset from the start of $B[j]$)
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

![Diagram of blocks]

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** Compute the predecessor of $x$ in $B[i]$

(suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:

Compute $j = \text{predecessor}(i)$ in $C$

Compute the predecessor of $x$ in $B[j]$

(suitably adjusting the offset from the start of $B[j]$)
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Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

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**Step 3** If $x$ has no predecessor in $B[i]$:
   - Compute $j = \text{predecessor}(i)$ in $C$
   - Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$)
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To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

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(suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:  
Compute $j = \text{predecessor}(i)$ in $C'$

Compute the predecessor of $x$ in $B[j]$

(suitably adjusting the offset from the start of $B[j]$)
Attempt 3: Recursion

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To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

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(suitably adjusting the offset from the start of $B[i]$)

**Step 3** If $x$ has no predecessor in $B[i]$:

Compute $j = \text{predecessor}(i)$ in $C$

Compute the predecessor of $x$ in $B[j]$

(suitably adjusting the offset from the start of $B[j]$)
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**Step 1** Determine which $B[i]$ the element $x$ belongs in.

**Step 2** Compute the predecessor of $x$ in $B[i]$ (suitably adjusting the offset from the start of $B[i]$).

**Step 3** If $x$ has no predecessor in $B[i]$: Compute $j = \text{predecessor}(i)$ in $C$.

Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$).
Attempt 3: Recursion

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Compute the predecessor of $x$ in $B[j]$

(suitably adjusting the offset from the start of $B[j]$)
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

The operations lookup, delete and successor can all also be defined in a similar, recursive manner.
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

The operations **lookup**, **delete** and **successor** can all also be defined in a similar, recursive manner.

How efficient are the operations?
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements

![Diagram of blocks](image)

To perform $\text{add}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in
   (this takes $O(1)$ time with a little bit twiddling)

**Step 2** If $B[i]$ is empty, add $i$ to $C$

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**Step 2** Compute the predecessor of $x$ in $B[i]$ (suitably adjusting the offset from the start of $B[i]$).

**Step 3** If $x$ has no predecessor in $B[i]$:

1. Compute $j = \text{predecessor}(i)$ in $C$.
2. Compute the predecessor of $x$ in $B[j]$ (suitably adjusting the offset from the start of $B[j]$).

predecessor makes up to three recursive calls!
**Attempt 3: Recursion**

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

The operations `lookup`, `delete` and `successor` can all also be defined in a similar, recursive manner.

How efficient are the operations?
Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

The operations lookup, delete and successor can all also be defined in a similar, recursive manner.

How efficient are the operations?

The add operation makes up to two recursive calls and the predecessor operation makes up to three
Attempt 3: Recursion

Split the universe \( U \) into \( \sqrt{u} \) blocks each associated with \( \sqrt{u} \) elements.

The operations **lookup**, **delete** and **successor** can all also be defined in a similar, recursive manner.

How efficient are the operations?

The **add** operation makes up to two recursive calls and the **predecessor** operation makes up to three.

Each recursive call could in turn make multiple recursive calls...
Attempt 3: Recursion

Split the universe $U$ into $\sqrt{u}$ blocks each associated with $\sqrt{u}$ elements.

The operations **lookup**, **delete** and **successor** can all also be defined in a similar, **recursive** manner.

How efficient are the operations?

The **add** operation makes up to two recursive calls and the **predecessor** operation makes up to three.

Each recursive call could in turn make multiple recursive calls... 

*This could get out of hand!*
A closer look at predecessor

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in.

**Step 2** Compute the predecessor of $x$ in $B[i]$.

**Step 3** If $x$ has no predecessor in $B[i]$:
- Compute $j = \text{predecessor}(i)$ in $C$.
- Return the predecessor of $x$ in $B[j]$. 

![Diagram showing the process of finding the predecessor of $x$.]
A closer look at predecessor

Observation 1: if $x$ has a predecessor in $B[i]$ we only make one recursive call

To perform $\text{predecessor}(x)$:

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   - Return the predecessor of $x$ in $B[j]$
A closer look at predecessor

**Observation 1:** if \( x \) has a predecessor in \( B[i] \) we only make one recursive call

\[ C \]

\[ \sqrt{u} \]

\( x \) has a predecessor in \( B[i] \) iff \( x \geq \text{the minimum in } B[i] \)

To perform \( \text{predecessor}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** Compute the predecessor of \( x \) in \( B[i] \)

**Step 3** If \( x \) has no predecessor in \( B[i] \):  
Compute \( j = \text{predecessor}(i) \) in \( C \)  
Return the predecessor of \( x \) in \( B[j] \)
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Compute $j = \text{predecessor}(i)$ in $C$
Return the predecessor of $x$ in $B[j]$
A closer look at predecessor

**Observation 1:** if $x$ has a predecessor in $B[i]$ we only make one recursive call

$x$ has a predecessor in $B[i]$ iff $x \geq$ the minimum in $B[i]$

To perform \texttt{predecessor}(x):

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** Compute the predecessor of $x$ in $B[i]$

**Step 3** If $x <$ the minimum in $B[i]$: 
Compute $j = \text{predecessor}(i)$ in $C$
Return the predecessor of $x$ in $B[j]$
A closer look at predecessor

**Observation 1:** if $x$ has a predecessor in $B[i]$ we only make one recursive call

$x$ has a predecessor in $B[i]$ iff $x \geq$ the minimum in $B[i]$

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**Step 2** If $x \geq$ the minimum in $B[i]$:
- Return the predecessor of $x$ in $B[i]$

**Step 3** If $x <$ the minimum in $B[i]$:
- Compute $j = \text{predecessor}(i)$ in $C$
- Return the predecessor of $x$ in $B[j]$
A closer look at predecessor

**Observation 1:** if \( x \) has a predecessor in \( B[i] \) we only make one recursive call.

\[
\begin{align*}
C & \quad \sqrt{u} \\
B[1] & \quad \sqrt{u} \\
B[2] & \quad \sqrt{u} \\
B[3] & \quad \sqrt{u} \\
\end{align*}
\]

\( x \) has a predecessor in \( B[i] \) iff \( x \geq \text{the minimum in } B[i] \)

To perform \texttt{predecessor}(\( x \)):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in.

**Step 2** If \( x \geq \text{the minimum in } B[i] \):

Return the predecessor of \( x \) in \( B[i] \).

**Step 3** If \( x < \text{the minimum in } B[i] \):

Compute \( j = \text{predecessor}(i) \) in \( C \).

Return the predecessor of \( x \) in \( B[j] \).

Now we make at most two recursive calls.
A closer look at predecessor

**Observation 1:** if \( x \) has a predecessor in \( B[i] \) we only make one recursive call

\[
x \text{ has a predecessor in } B[i] \iff x \geq \text{the minimum in } B[i]
\]

To perform \( \text{predecessor}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( x \geq \text{the minimum in } B[i] \):
- Return the predecessor of \( x \) in \( B[i] \)

**Step 3** If \( x < \text{the minimum in } B[i] \):
- Compute \( j = \text{predecessor}(i) \) in \( C \)
- Return the predecessor of \( x \) in \( B[j] \)

Now we make at most two recursive calls (ignoring finding the minimum)
A closer look at predecessor

To perform \texttt{predecessor}(x):

\textbf{Step 1} Determine which \(B[i]\) the element \(x\) belongs in

\textbf{Step 2} If \(x \geq \text{the minimum in } B[i]\):

Return the \texttt{predecessor} of \(x\) in \(B[i]\)

\textbf{Step 3} If \(x < \text{the minimum in } B[i]\):

Compute \(j = \text{predecessor}(i)\) in \(C\)

Return the \texttt{predecessor} of \(x\) in \(B[j]\)
A closer look at predecessor

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** If $x \geq$ the minimum in $B[i]$:
- Return the predecessor of $x$ in $B[i]$

**Step 3** If $x <$ the minimum in $B[i]$:
- Compute $j = \text{predecessor}(i)$ in $C$
- Return the predecessor of $x$ in $B[j]$

we need to get rid of one of these recursive calls
A closer look at predecessor

To perform predecessor(x):

**Step 1** Determine which B[i] the element x belongs in

**Step 2** If \( x \geq \) the minimum in \( B[i] \):
- Return the predecessor of x in \( B[i] \)

**Step 3** If \( x < \) the minimum in \( B[i] \):
- Compute \( j = \) predecessor(i) in C
- Return the predecessor of x in \( B[j] \)

we need to get rid of one of these recursive calls
A closer look at predecessor

Observation 2: In Step 3, the predecessor of $x$ in $B[j]$ is the maximum in $B[j]$

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** If $x \geq$ the minimum in $B[i]$:  
Return the predecessor of $x$ in $B[i]$

**Step 3** If $x <$ the minimum in $B[i]$:  
Compute $j = \text{predecessor}(i)$ in $C$  
Return the predecessor of $x$ in $B[j]$

We need to get rid of one of these recursive calls.
A closer look at predecessor

Observation 2: In Step 3, the predecessor of $x$ in $B[j]$ is the maximum in $B[j]$

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** If $x \geq$ the minimum in $B[i]$:  
Return the predecessor of $x$ in $B[i]$

**Step 3** If $x <$ the minimum in $B[i]$:  
Compute $j = \text{predecessor}(i)$ in $C$  
Return the predecessor of $x$ in $B[j]$
A closer look at predecessor

Observation 2: In Step 3, the predecessor of $x$ in $B[j]$ is the maximum in $B[j]$

To perform $\text{predecessor}(x)$:

**Step 1** Determine which $B[i]$ the element $x$ belongs in

**Step 2** If $x \geq$ the minimum in $B[i]$:
   Return the predecessor of $x$ in $B[i]$

**Step 3** If $x <$ the minimum in $B[i]$:
   Compute $j = \text{predecessor}(i)$ in $C$
   Return the maximum in $B[j]$
A closer look at predecessor

Observation 2: In **Step 3**, the predecessor of \( x \) in \( B[j] \) is the maximum in \( B[j] \)

To perform \( \text{predecessor}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( x \geq \) the minimum in \( B[i] \):
   - Return the predecessor of \( x \) in \( B[i] \)

**Step 3** If \( x < \) the minimum in \( B[i] \):
   - Compute \( j = \text{predecessor}(i) \) in \( C \)
   - Return the maximum in \( B[j] \)

Now we make exactly one recursive call (ignoring finding the min/max)
Finally: van Emde Boas Trees

So that we can find the min/max quickly we store them separately...
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Remember that each $B[i]$ and $C$ are also vEB (van Emde Boas) trees each over the universe $\{1, 2, 3, \ldots \sqrt{u}\}$
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There is one more important thing, the minimum is not also stored in $B[i]$ this allows us to avoid making multiple recursive calls when adding an element
Another look at add

To perform \( \text{add}(x) \):

**Step 1** Determine which \( B[i] \) the element \( x \) belongs in

**Step 2** If \( B[i] \) is empty, add \( i \) to \( C \)  
and set the min and max in \( B[i] \) to \( x \) *(adjusting the offset)*

**Step 3** If \( B[i] \) is not empty, add \( x \) to \( B[i] \)
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the min is only stored here
Another look at add

To perform add\((x)\):

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---

We make one recursive call.

---

This is not recursive.
Another look at `add`

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Now we always make exactly one recursive call but what happens when the min/max change?

the min is only stored here


\[ \min \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \sqrt{u} \quad \max \]

\[ u \]

min

\[ 37 \quad 483 \]
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Now we always make exactly one recursive call but what happens when the \text{min}/\text{max} change?
Another look at add

To perform add\((x)\):

**Step 0** If \(x < \text{min}\) then swap \(x\) and \(\text{min}\)

**Step 1** Determine which \(B[i]\) the element \(x\) belongs in

**Step 2** If \(B[i]\) is empty, add \(i\) to \(C\)

and set the \(\text{min}\) and \(\text{max}\) in \(B[i]\) to \(x\) (*adjusting the offset*)

**Step 3** If \(B[i]\) is not empty, add \(x\) to \(B[i]\)

**Step 4** Update the \(\text{max}\)
the min is only stored here
We have seen that the operations \textit{add} and \textit{predecessor} can be defined so that they make only one recursive call.
Time Complexity

We have seen that the operations add and predecessor can be defined so that they make only one recursive call.

The operations lookup, delete and successor can all also be defined in a similar, recursive manner so that they make only one recursive call.
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How long do the operations take?
the min is only stored here

Time Complexity
Let $T(u)$ be the time complexity of the add operation

(where $u$ is the universe size)
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Using substitution and the master method you can show that… 

$$T(u) = O(\log \log u)$$
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Using substitution and the master method you can show that... $T(u) = O(\log \log u)$

this holds for all the operations
Space Complexity

the min is only stored here

\[ C \]

\[ B[1] \]

\[ B[2] \]

\[ B[3] \]

\[ B[\sqrt{u}] \]

min \[ \sqrt{u} \] \[ \sqrt{u} \] \[ \sqrt{u} \] \[ \sqrt{u} \] \[ \sqrt{u} \] \[ \sqrt{u} \] \[ \sqrt{u} \] max

\[ u \]
Space Complexity

Let $Z(u)$ be the space used by a vEB tree over a universe of size $u$. 

---

**Diagram:**

- $C$ is a node in the vEB tree.
- $B[\sqrt{u}]$ is a branch for larger values.
- The minimum value is stored in $B[1]$.
- The maximum value is stored in $B[3]$.
- The space complexity is $\sqrt{u}$ for each branch.

---

**Note:** The min is only stored here.
Let $Z(u)$ be the space used by a vEB tree over a universe of size $u$. We have that,

$$Z(u) = (\sqrt{u} + 1) \cdot Z(\sqrt{u}) + O(1)$$
Let $Z(u)$ be the space used by a vEB tree over a universe of size $u$.

We have that, $Z(u) = (\sqrt{u} + 1) \cdot Z(\sqrt{u}) + O(1)$

If you solve this you get that... $Z(u) = O(u)$
van Emde Boas Trees

The van Emde Boas (vEB) tree stores a set $S$ of integer keys from a universe $U = \{1, 2, 3, 4 \ldots u\}$ (i.e. $u = |U|$). Five operations are supported:

- **add($x$)**: Insert the integer $x$ into $S$ (where $x \in U$)
- **lookup($x$)**: Return yes if $x$ is in $S$, or no otherwise.
- **delete($x$)**: Remove $x$ from $S$
- **predecessor($k$)**: Return the largest integer $x$ in $S$ such that $x \leq k$
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All operations take $O(\log \log u)$ worst case time and the space used is $O(u)$.
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The space can be improved to $O(n)$ using hashing (see y-fast trees)