Advanced Algorithms – COMS31900

Hashing part two
Static Perfect Hashing

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Dictionaries and Hashing recap

A **dynamic dictionary** stores \((key, value)\)-pairs and supports:

- `add(key, value)`, `lookup(key)` (which returns `value`) and `delete(key)`

- Universe \(U\) of \(u\) keys.
- Hash table \(T\) of size \(m \geq n\).
- Collisions were fixed by **chaining** (building linked lists)

A **hash function** maps a key \(x\) to position \(h(x)\)
- i.e. \(T[h(x)] = (key, value)\).

\(n\) arbitrary operations arrive online, one at a time.
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A set \(H\) of hash functions is **weakly universal** if for any two keys \(x, y \in U\) (with \(x \neq y\)),

\[
\Pr(h(x) = h(y)) \leq \frac{1}{m}
\]

\((h\ \text{is picked uniformly at random from } H)\)
Dictionaries and Hashing recap

- **A dynamic dictionary** stores \((\text{key}, \text{value})\)-pairs and supports:
  
  \[
  \text{add}(\text{key}, \text{value}), \text{lookup}(\text{key}) \text{ (which returns \text{value}) and delete(\text{key})}
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Using weakly universal hashing:

For any \(n\) operations, the expected run-time is \(O(1)\) per operation.
Dictionaries and Hashing recap

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Using weakly universal hashing:

For any \(n\) operations, the expected run-time is \(O(1)\) per operation.

But this doesn’t tell us much about the worst-case behaviour
A static dictionary stores \((key, value)\)-pairs and supports:

- \(\text{lookup}(key)\) (which returns \(value\)) - no inserts or deletes are allowed.

We are given \(n\) different \((key, value)\)-pairs and want to pick a good \(h\).

Universe \(U\) of \(u\) keys.

Hash table \(T\) of size \(m \geq n\).

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Static Dictionaries and Perfect hashing

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We are given \(n\) different \((key, value)\)-pairs and want to pick a good \(h\).

**Theorem**

The FKS hashing scheme:

- Has no collisions
- Every lookup takes \(O(1)\) worst-case time,
- Uses \(O(n)\) space,
- Can be built in \(O(n)\) expected time.
Static Dictionaries and Perfect hashing

- A **static dictionary** stores \((key, value)\)-pairs and supports:
  
  \(\text{lookup}(key)\) (which returns \(value\)) - no inserts or deletes are allowed

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**Theorem**

- The FKS hashing scheme:
  - Has no collisions
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The rest of this lecture is devoted to the FKS scheme.
A static dictionary stores \((key, value)\)-pairs and supports:

\[\text{lookup}(key) \text{ (which returns value) - no inserts or deletes are allowed}\]

The rest of this lecture is devoted to the FKS scheme

The construction is based on weak universal hashing
A static dictionary stores \((key, value)\)-pairs and supports:

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\text{lookup}(\text{key}) \text{ (which returns \text{value}) - no inserts or deletes are allowed}
\]

A hash function maps a key \(x\) to position \(h(x)\) - i.e \(T[h(x)] = (\text{key}, \text{value})\).

we are given \(n\) different \((\text{key}, \text{value})\)-pairs and want to pick a good \(h\)

**Theorem**

The FKS hashing scheme:
- Has no collisions
- Every lookup takes \(O(1)\) worst-case time,
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The rest of this lecture is devoted to the FKS scheme

The construction is based on weak universal hashing

(with an \(O(1)\) time hash function)
A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U \ (x \neq y)$,

$$\Pr(h(x) = h(y)) \leq \frac{1}{m} \quad \text{where } h \text{ is picked uniformly at random from } H$$
A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U$ ($x \neq y$),

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where $h$ is picked uniformly at random from $H$.

**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function.
A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U$ ($x \neq y$),

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where $h$ is picked uniformly at random from $H$.

**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function.

*(where any $h(x)$ can be computed in $O(1)$ time)*
Perfect hashing - a first attempt

A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U \ (x \neq y)$,

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\Pr \left( h(x) = h(y) \right) \leq \frac{1}{m}
$$

where $h$ is picked uniformly at random from $H$

**Step 1:** Insert everything into a hash table of size $m = n$
using a weakly universal hash function
Perfect hashing - a first attempt

A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U$ ($x \neq y$),

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---

Step 1: Insert everything into a hash table of size $m = n$ using a weakly universal hash function

Step 2: Check for collisions
Perfect hashing - a first attempt

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where $h$ is picked uniformly at random from $H$

**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** Profit!
Perfect hashing - a first attempt

A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U$ ($x \neq y$),

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**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if necessary*
Perfect hashing - a first attempt

A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U \ (x \neq y)$,

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**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** Repeat if necessary

How many collisions do we get on average?
Perfect hashing - a first attempt

A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U$ ($x \neq y$),

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where $h$ is picked uniformly at random from $H$.

**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function.

**Step 2:** Check for collisions.

**Step 3:** Repeat if necessary.

How many collisions do we get on average?

The expected number of collisions is given by:

$$E(C) = E\left( \sum_{x, y \in T, x < y} I_{x, y} \right)$$

where indicator random variable $I_{x, y} = 1$ iff $h(x) = h(y)$.

$n$
Perfect hashing - a first attempt

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---

**Step 1:** Insert everything into a hash table of size $m = n$

using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if necessary*

---

How many collisions do we get on average?

number of collisions

$$\mathbb{E}(C') = \mathbb{E}( \sum_{x,y \in T, x < y} I_{x,y} ) = \sum_{x,y \in T, x < y} \mathbb{E}(I_{x,y})$$

where indicator random variable $I_{x,y} = 1$ iff $h(x) = h(y)$. 
Perfect hashing - a first attempt

A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U$ ($x \neq y$),

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**Linearity of Expectation**

Let $Y_1, Y_2, \ldots, Y_k$ be $k$ random variables. Then

$$\mathbb{E}
\left(
\sum_{i=1}^{k} Y_i
\right)
= \sum_{i=1}^{k} \mathbb{E}(Y_i)$$

Number of collisions

$$\mathbb{E}(C) = \mathbb{E}
\left(
\sum_{x,y \in T, x < y} I_{x,y}
\right)
= \sum_{x,y \in T, x < y} \mathbb{E}(I_{x,y})$$

where indicator random variable $I_{x,y} = 1$ iff $h(x) = h(y)$.
Perfect hashing - a first attempt

A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U$ ($x \neq y$),

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**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if necessary*

---

**How many collisions do we get on average?**

The expected number of collisions can be calculated as follows:

$$E(C) = E\left( \sum_{x,y \in T, x < y} I_{x,y} \right) = \sum_{x,y \in T, x < y} E(I_{x,y})$$

where indicator random variable $I_{x,y} = 1$ iff $h(x) = h(y)$. 

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The explanation on the slide states that a set $H$ of hash functions is weakly universal if for any two distinct keys $x, y$ from a universe $U$, the probability that they hash to the same value is at most $1/m$. This property is leveraged in the first attempt at perfect hashing, where items are inserted into a hash table of size $m = n$ using a weakly universal hash function. Collisions are then checked, and if necessary, the process is repeated.
Perfect hashing - a first attempt

A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U \ (x \neq y)$,

$$\Pr (h(x) = h(y)) \leq \frac{1}{m}$$

where $h$ is picked uniformly at random from $H$

---

**Step 1:** Insert everything into a hash table of size $m = n$

using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if necessary*

---

**How many collisions do we get on average?**

Number of collisions

Linearity of expectation

$$\mathbb{E}(C) = \mathbb{E} \left( \sum_{x, y \in T, x < y} I_{x, y} \right) = \sum_{x, y \in T, x < y} \mathbb{E}(I_{x, y}) \leq \sum_{x, y \in T, x < y} \frac{1}{m}$$

where indicator random variable $I_{x, y} = 1$ iff $h(x) = h(y)$. 
A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U \ (x \neq y)$,

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**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function.

**Step 2:** Check for collisions.

**Step 3:** *Repeat if necessary*

**How many collisions do we get on average?**

$$E(C) = E\left( \sum_{x,y \in T, x < y} I_{x,y} \right) = \sum_{x,y \in T, x < y} E(I_{x,y}) \leq \sum_{x,y \in T, x < y} \frac{1}{m}$$

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Perfect hashing - a first attempt

A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U$ ($x \neq y$),

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**Step 1:** Insert everything into a hash table of size $m = n$
using a weakly universal hash function

**Step 2:** Check for collisions

By the definition of expectation...

$$E(I_{x,y}) = 1 \cdot \Pr(I_{x,y} = 1) + 0 \cdot \Pr(I_{x,y} = 0) \leq \frac{1}{m}$$

number of collisions \hspace{2cm} linearity of expectation

$$E(C) = E\left( \sum_{x,y \in T, x < y} I_{x,y} \right) = \sum_{x,y \in T, x < y} E(I_{x,y}) \leq \sum_{x,y \in T, x < y} \frac{1}{m}$$

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Perfect hashing - a first attempt

A set \( H \) of hash functions is **weakly universal** if for any two keys \( x, y \in U \) \((x \neq y)\),

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**Step 1:** Insert everything into a hash table of size \( m = n \) using a weakly universal hash function.

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**Step 3:** *Repeat if necessary*

**How many collisions do we get on average?**

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Perfect hashing - a first attempt

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**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** Repeat if necessary

**How many collisions do we get on average?**

The expected number of collisions is given by:

$$\mathbb{E}(C) = \mathbb{E}\left( \sum_{x,y \in T, x < y} I_{x,y} \right) = \sum_{x,y \in T, x < y} \mathbb{E}(I_{x,y}) \leq \sum_{x,y \in T, x < y} \frac{1}{m} = \binom{n}{2} \cdot \frac{1}{m}$$

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**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function.

**Step 2:** Check for collisions.

**Step 3:** Repeat if necessary.

*How many collisions do we get on average?*

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\mathbb{E}(C) = \mathbb{E}
\left(\sum_{x,y \in T, x < y} I_{x,y}\right)
= \sum_{x,y \in T, x < y} \mathbb{E}(I_{x,y})
\leq \sum_{x,y \in T, x < y} \frac{1}{m}
= \binom{n}{2} \cdot \frac{1}{m}
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where indicator random variable $I_{x,y} = 1$ iff $h(x) = h(y)$. 
Perfect hashing - a first attempt

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**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function.

**Step 2:** Check for collisions.

**Step 3:** Repeat if necessary.

**How many collisions do we get on average?**

$$E(C') = E\left(\sum_{x, y \in T, x < y} I_{x, y}\right) = \sum_{x, y \in T, x < y} E(I_{x, y}) \leq \sum_{x, y \in T, x < y} \frac{1}{m} = \binom{n}{2} \cdot \frac{1}{m} \leq \frac{n^2}{2m}$$

where indicator random variable $I_{x, y} = 1$ iff $h(x) = h(y)$. 
Perfect hashing - a first attempt

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**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if necessary*

**How many collisions do we get on average?**

The expected number of collisions is given by:

$$E(C) = E\left( \sum_{x,y \in T, x < y} I_{x,y} \right) = \sum_{x,y \in T, x < y} E(I_{x,y}) \leq \sum_{x,y \in T, x < y} \frac{1}{m} = \binom{n}{2} \cdot \frac{1}{m} \leq \frac{n^2}{2m} \leq \frac{n}{2}.$$

where indicator random variable $I_{x,y} = 1$ iff $h(x) = h(y)$. 
A set $H$ of hash functions is \textbf{weakly universal} if for any two keys $x, y \in U$ ($x \neq y$),

$$\Pr(h(x) = h(y)) \leq \frac{1}{m}$$

where $h$ is picked uniformly at random from $H$.

\textbf{Step 1:} Insert everything into a hash table of size $m = n^2$ using a weakly universal hash function.

\textbf{Step 2:} Check for collisions.

\textbf{Step 3:} \textit{Repeat if necessary}. 

Perfect hashing - a second attempt
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A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U$ ($x \neq y$),

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where $h$ is picked uniformly at random from $H$.

**Step 1:** Insert everything into a hash table of size $m = n^2$ using a weakly universal hash function.

**Step 2:** Check for collisions.

**Step 3:** *Repeat if necessary*

*How many collisions do we get on average?*
A set $H$ of hash functions is **weakly universal** if for any two keys $x, y \in U$ ($x \neq y$),

$$\Pr (h(x) = h(y)) \leq \frac{1}{m}$$

where $h$ is picked uniformly at random from $H$.

**Step 1:** Insert everything into a hash table of size $m = n^2$

Using a weakly universal hash function.

**Step 2:** Check for collisions

**Step 3:** Repeat if necessary

**How many collisions do we get on average?**

Number of collisions \( \sum_{x,y \in T, x<y} I_{x,y} \)

Linearity of expectation \( \sum_{x,y \in T, x<y} \mathbb{E}(I_{x,y}) \)

Definition of expectation \( \sum_{x,y \in T, x<y} \frac{1}{m} \)

\( \leq n^2 / 2 \)

where indicator random variable $I_{x,y} = 1$ iff $h(x) = h(y)$. 
Perfect hashing - a second attempt

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where indicator random variable $I_{x, y} = 1$ iff $h(x) = h(y)$. 

The linearity of expectation is used to simplify the expectation of the sum of indicator variables.
Perfect hashing - a second attempt

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where $h$ is picked uniformly at random from $H$.

**Step 1:** Insert everything into a hash table of size $m = n^2$ using a weakly universal hash function.

**Step 2:** Check for collisions.

**Step 3:** *Repeat if necessary.*

How many collisions do we get on average?

$$E(C) = E\left( \sum_{x,y \in T, x < y} I_{x,y} \right) = \sum_{x,y \in T, x < y} E(I_{x,y}) \leq \sum_{x,y \in T, x < y} \frac{1}{m} = \frac{n^2}{2} \cdot \frac{1}{m} \leq \frac{n^2}{2m} \leq \frac{1}{2}$$

where indicator random variable $I_{x,y} = 1$ iff $h(x) = h(y)$. *much better!*)
Perfect hashing - a second attempt

A set $H$ of hash functions is weakly universal if for any two keys $x, y \in U$ ($x \neq y$),

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where $h$ is picked uniformly at random from $H$

---

**Step 1:** Insert everything into a hash table of size $m = n^2$

**Step 2:** Check for collisions

**Step 3:** Repeat if necessary

(except we cheated)

How many collisions do we get on average?

$$E(C) = E\left( \sum_{x,y \in T, x<y} I_{x,y} \right) = \sum_{x,y \in T, x<y} E(I_{x,y}) \leq \sum_{x,y \in T, x<y} \frac{1}{m} = \frac{n}{2} \cdot \frac{1}{m} \leq \frac{n^2}{2m} \leq \frac{1}{2}$$

where indicator random variable $I_{x,y} = 1$ iff $h(x) = h(y)$. Much better!
Expected construction time

**Step 1:** Insert everything into a hash table of size $m = n^2$
using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** Repeat if there was a collision
Expected construction time

Step 1: Insert everything into a hash table of size \( m = n^2 \)
using a weakly universal hash function

Step 2: Check for collisions

Step 3: Repeat if there was a collision

How many times do we repeat on average?
Expected construction time

**Step 1:** Insert everything into a hash table of size $m = n^2$
using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if there was a collision*

*How many times do we repeat on average?*

The expected number of collisions: $\mathbb{E}(C) \leq \frac{1}{2}$

The probability of at least one collision: $\Pr(C \geq 1) \leq \frac{1}{2}$
Expected construction time

Step 1: Insert everything into a hash table of size $m = n^2$

Step 2: Check for collisions

Step 3: Repeat if there was a collision

How many times do we repeat on average?

The expected number of collisions: $\mathbb{E}(C) \leq \frac{1}{2}$

The probability of at least one collision: $\Pr(C \geq 1) \leq \frac{1}{2}$

Markov’s inequality

If $X$ is a non-negative r.v., then for all $a > 0$,

$$\Pr(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$
Expected construction time

**Step 1:** Insert everything into a hash table of size $m = n^2$

using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if there was a collision*

How many times do we repeat on average?

The expected number of collisions: $\mathbb{E}(C) \leq \frac{1}{2}$

The probability of at least one collision: $\Pr(C \geq 1) \leq \frac{1}{2}$

Markov’s inequality
Expected construction time

**Step 1:** Insert everything into a hash table of size $m = n^2$ using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if there was a collision*

*How many times do we repeat on average?*

The expected number of collisions: $\mathbb{E}(C') \leq \frac{1}{2}$

The probability of at least one collision: $\Pr(C \geq 1) \leq \frac{1}{2}$

The probability of zero collisions is at least $\frac{1}{2}$
Expected construction time

**Step 1:** Insert everything into a hash table of size \( m = n^2 \) using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** Repeat if there was a collision

**How many times do we repeat on average?**

The expected number of collisions: \( \mathbb{E}(C) \leq \frac{1}{2} \)  

The probability of at least one collision: \( \Pr(C \geq 1) \leq \frac{1}{2} \)

The probability of zero collisions is at least \( \frac{1}{2} \)  

* i.e. at least as good as tossing a heads on a fair coin

Markov’s inequality
Expected construction time

Step 1: Insert everything into a hash table of size $m = n^2$ using a weakly universal hash function

Step 2: Check for collisions

Step 3: Repeat if there was a collision

How many times do we repeat on average?

The expected number of collisions: $\mathbb{E}(C') \leq \frac{1}{2}$

The probability of at least one collision: $\Pr(C \geq 1) \leq \frac{1}{2}$

The probability of zero collisions is at least $\frac{1}{2}$

i.e. at least as good as tossing a heads on a fair coin

$\mathbb{E}(\text{runs}) \leq \mathbb{E}(\text{coin tosses to get a heads}) = 2$
**Expected construction time**

**Step 1:** Insert everything into a hash table of size $m = n^2$ using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if there was a collision*

---

**How many times do we repeat on average?**

The expected number of collisions: $\mathbb{E}(C) \leq \frac{1}{2}$  

Markov’s inequality

The probability of at least one collision: $\Pr(C \geq 1) \leq \frac{1}{2}$

The probability of zero collisions is at least $\frac{1}{2}$  

*i.e. at least as good as tossing a heads on a fair coin*

$\mathbb{E}(\text{runs}) \leq \mathbb{E}(\text{coin tosses to get a heads}) = 2$

$\mathbb{E}(\text{construction time}) = O(m) \cdot \mathbb{E}(\text{runs}) = O(m) = O(n^2)$
Expected construction time

**Step 1:** Insert everything into a hash table of size $m = n^2$
using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** Repeat if there was a collision

*How many times do we repeat on average?*

The expected number of collisions: $\mathbb{E}(C) \leq \frac{1}{2}$

The probability of at least one collision: $\Pr(C \geq 1) \leq \frac{1}{2}$

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$\mathbb{E}(\text{construction time}) = O(m) \cdot \mathbb{E}(\text{runs}) = O(m) = O(n^2)$

... and then the look-up time is always $O(1)$
Expected construction time

**Step 1:** Insert everything into a hash table of size $m = n^2$
using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if there was a collision*

---

**How many times do we repeat on average?**

The expected number of collisions: $E(C) \leq \frac{1}{2}$ **Markov’s inequality**

The probability of at least one collision: $\Pr(C \geq 1) \leq \frac{1}{2}$

The probability of zero collisions is at least $\frac{1}{2}$

*i.e. at least as good as tossing a heads on a fair coin*

$E(\text{runs}) \leq E(\text{coin tosses to get a heads}) = 2$

$E(\text{construction time}) = O(m) \cdot E(\text{runs}) = O(m) = O(n^2)$

... and then the look-up time is always $O(1)$

*(because any $h(x)$ can be computed in $O(1)$ time)*
Expected construction time

**Step 1:** Insert everything into a hash table of size \( m = n \) using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if there are more than \( n \) collisions*
Expected construction time

**Step 1:** Insert everything into a hash table of size $m = n$
using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if there are more than $n$ collisions*

This looks rubbish but it will be useful in a bit!
Expected construction time

**Step 1:** Insert everything into a hash table of size $m = n$ using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** Repeat if there are more than $n$ collisions

This looks rubbish but it will be useful in a bit!

*How many times do we repeat on average?*

The expected number of collisions: $\mathbb{E}(C) \leq \frac{n}{2}$

The probability of at least $n$ collisions: $\Pr(C \geq n) \leq \frac{1}{2}$
Expected construction time

Step 1: Insert everything into a hash table of size $m = n$

Markov's inequality
If $X$ is a non-negative r.v., then for all $a > 0$,\

\[
\Pr(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.
\]

The expected number of collisions: $\mathbb{E}(C) \leq \frac{n}{2}$

The probability of at least $n$ collisions: $\Pr(C \geq n) \leq \frac{1}{2}$ (where $a = n$)

This looks rubbish but it will be useful in a bit!
Expected construction time

**Step 1:** Insert everything into a hash table of size \( m = n \) using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** *Repeat if there are more than \( n \) collisions*

This looks rubbish but it will be useful in a bit!

*How many times do we repeat on average?*

The expected number of collisions: \( \mathbb{E}(C) \leq \frac{n}{2} \)

The probability of at least \( n \) collisions: \( \Pr(C \geq n) \leq \frac{1}{2} \)
Expected construction time

Step 1: Insert everything into a hash table of size $m = n$ using a weakly universal hash function

Step 2: Check for collisions

Step 3: Repeat if there are more than $n$ collisions

How many times do we repeat on average?

The expected number of collisions: $\mathbb{E}(C) \leq \frac{n}{2}$

The probability of at least $n$ collisions: $\Pr(C \geq n) \leq \frac{1}{2}$

The probability of at most $n$ collisions is at least $\frac{1}{2}$

i.e. at least as good as tossing a heads on a fair coin

$\mathbb{E}(\text{runs}) \leq \mathbb{E}(\text{coin tosses to get a heads}) = 2$

$\mathbb{E}(\text{construction time}) = O(m) \cdot \mathbb{E}(\text{runs}) = O(m) = O(n)$

This looks rubbish but it will be useful in a bit!
Expected construction time

**Step 1:** Insert everything into a hash table of size \( m = n \) using a weakly universal hash function

**Step 2:** Check for collisions

**Step 3:** Repeat if there are more than \( n \) collisions

This looks rubbish but it will be useful in a bit!

**How many times do we repeat on average?**

The expected number of collisions: \( \mathbb{E}(C) \leq \frac{n}{2} \)

The probability of at least \( n \) collisions: \( \Pr(C \geq n) \leq \frac{1}{2} \)

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i.e. at least as good as tossing a heads on a fair coin

\[ \mathbb{E}(\text{runs}) \leq \mathbb{E}(\text{coin tosses to get a heads}) = 2 \]

\[ \mathbb{E}(\text{construction time}) = O(m) \cdot \mathbb{E}(\text{runs}) = O(m) = O(n) \]

... but the look-up time could be rubbish (lots of collisions)
Perfect hashing - attempt three

Step 1: Insert everything into a hash table, $T$, of size $n$ using a weakly universal hash function, $h$. 
Perfect hashing - attempt three

**Step 1:** Insert everything into a hash table, $T$, of size $n$
using a weakly universal hash function, $h$

Let $n_i$ be the number of items in $T[i]$
Perfect hashing - attempt three

Step 1: Insert everything into a hash table, $T$, of size $n$
using a weakly universal hash function, $h$

Let $n_i$ be the number of items in $T[i]$

$n_1 = 2$

$n_5 = 2$

$n_8 = 3$
Perfect hashing - attempt three

**Step 1:** Insert everything into a hash table, $T$, of size $n$ using a weakly universal hash function, $h$ 

...but don’t use chaining

Let $n_i$ be the number of items in $T[i]$

**Step 2:** The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$

*using another weakly universal hash function denoted $h_i$ (there is one for each $i$)*
Perfect hashing - attempt three

Step 1: Insert everything into a hash table, \( T \), of size \( n \) using a weakly universal hash function, \( h \)

\[ \ldots \text{but don't use chaining} \]

Step 2: The \( n_i \) items in \( T[i] \) are inserted into another hash table \( T_i \) of size \( n_i^2 \)

using another weakly universal hash function denoted \( h_i \) (there is one for each \( i \))

Let \( n_i \) be the number of items in \( T[i] \)
Perfect hashing - attempt three

**Step 1:** Insert everything into a hash table, $T$, of size $n$
using a weakly universal hash function, $h$

... but don't use chaining

Let $n_i$ be the number of items in $T[i]$

**Step 2:** The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$

using another weakly universal hash function
denoted $h_i$ (there is one for each $i$)

**Step 3** Immediately repeat a step if either

a) $T$ has more than $n$ collisions
b) some $T_i$ has a collision
Perfect hashing - attempt three

**Step 1:** Insert everything into a hash table, \( T \), of size \( n \) using a weakly universal hash function, \( h \) … but don’t use chaining

Let \( n_i \) be the number of items in \( T[i] \)

**Step 2:** The \( n_i \) items in \( T[i] \) are inserted into another hash table \( T_i \) of size \( n_{2i}^2 \) using another weakly universal hash function denoted \( h_i \) (there is one for each \( i \))

**Step 3** Immediately repeat a step if either
a) \( T \) has more than \( n \) collisions
b) some \( T_i \) has a collision

i.e. check (and if necessary rebuild) each table immediately after building it
Perfect hashing - attempt three

**Step 1:** Insert everything into a hash table, $T$, of size $n$
using a weakly universal hash function, $h$

... but don't use chaining

Let $n_i$ be the number of items in $T[i]$

**Step 2:** The $n_i$ items in $T[i]$ are inserted into
another hash table $T_i$ of size $n_i^2$

using another weakly universal hash function
denoted $h_i$ (there is one for each $i$)

(Step 3) Immediately repeat a step if either
a) $T$ has more than $n$ collisions
b) some $T_i$ has a collision
Perfect hashing - attempt three

**Step 1:** Insert everything into a hash table, $T$, of size $n$
using a weakly universal hash function, $h$

...but don’t use chaining

Let $n_i$ be the number of items in $T[i]$

**Step 2:** The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$

using another weakly universal hash function denoted $h_i$ (there is one for each $i$)

(Step 3) *Immediately repeat a step if either*

a) $T$ has more than $n$ collisions

b) some $T_i$ has a collision

The look-up time is always $O(1)$

1. Compute $i = h(x)$ ($x$ is the key)
2. Compute $j = h_i(x)$
3. The item is in $T_i[j]$
Perfect hashing - attempt three

**Step 1:** Insert everything into a hash table, $T$, of size $n$ using a weakly universal hash function, $h$ … but don’t use chaining

Let $n_i$ be the number of items in $T[i]$.

**Step 2:** The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$

using another weakly universal hash function denoted $h_i$ (there is one for each $i$)

(Step 3) *Immediately repeat a step if either*

a) $T$ has more than $n$ collisions

b) some $T_i$ has a collision

The look-up time is always $O(1)$

1. Compute $i = h(x)$ ($x$ is the key)
2. Compute $j = h_i(x)$
3. The item is in $T_i[j]$

Two questions remain:

What is the expected construction time?

What is the space usage?
Perfect Hashing - Space usage

Step 1: Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$

Step 2: The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$

(Step 3) Immediately repeat if either
a) $T$ has more than $n$ collisions
b) some $T_i$ has a collision

How much space does this use?
Perfect Hashing - Space usage

Step 1: Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$.

Step 2: The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$.

(Step 3) Immediately repeat if either
   a) $T$ has more than $n$ collisions
   b) some $T_i$ has a collision

How much space does this use?

The size of $T$ is $O(n)$
Perfect Hashing - Space usage

Step 1: Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$

Step 2: The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$

(Step 3) Immediately repeat if either
a) $T$ has more than $n$ collisions
b) some $T_i$ has a collision

How much space does this use?

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n_i^2)$
Perfect Hashing - Space usage

Step 1: Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$

Step 2: The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$

(Step 3) Immediately repeat if either
  a) $T$ has more than $n$ collisions
  b) some $T_i$ has a collision

How much space does this use?

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n_i^2)$

Storing $h_i$ uses $O(1)$ space
Perfect Hashing - Space usage

Step 1: Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$.

Step 2: The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$.

(Step 3) *Immediately repeat if either*

a) $T$ has more than $n$ collisions
b) some $T_i$ has a collision

**How much space does this use?**

The size of $T$ is $O(n)$.

The size of $T_i$ is $O(n_i^2)$.

Storing $h_i$ uses $O(1)$ space.

*So the total space is…*
Perfect Hashing - Space usage

**Step 1**: Insert everything into a hash table, $T$, of size $n$
using a weakly universal (w.u.) hash function, $h$

**Step 2**: The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$
of size $n_i^2$ using w.u hash function $h_i$

**(Step 3)** *Immediately repeat if either*

a) $T$ has more than $n$ collisions

b) some $T_i$ has a collision

---

*How much space does this use?*

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n_i^2)$

Storing $h_i$ uses $O(1)$ space

*So the total space is…*

$$O(n) + \sum_i O(n_i^2)$$
Perfect Hashing - Space usage

**Step 1:** Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$.

**Step 2:** The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$.

**Step 3)** Immediately repeat if either
- a) $T$ has more than $n$ collisions
- b) some $T_i$ has a collision

---

**How much space does this use?**

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n_i^2)$

Storing $h_i$ uses $O(1)$ space

*So the total space is…*

$$O(n) + \sum_i O(n_i^2) = O(n) + O\left(\sum_i n_i^2\right)$$
**Perfect Hashing - Space usage**

**Step 1:** Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$.

**Step 2:** The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$.

**Step 3** *Immediately repeat if either*

a) $T$ has more than $n$ collisions
b) some $T_i$ has a collision

---

**How much space does this use?**

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n_i^2)$

Storing $h_i$ uses $O(1)$ space

*So the total space is…*

$$O(n) + \sum_i O(n_i^2) = O(n) + O\left(\sum_i n_i^2\right)$$
Perfect Hashing - Space usage

**Step 1:** Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$.

**Step 2:** The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$.

How much space does this use?

(Step 3) Immediately repeat if either

- a) $T$ has more than $n$ collisions
- b) some $T_i$ has a collision

The size of $T$ is $O(n)$.

The size of $T_i$ is $O(n_i^2)$.

So the total space is...

Storing $h_i$ uses $O(1)$ space.

So the total space is...

$$O(n) + \sum_i O(n_i^2) = O(n) + O\left(\sum_i n_i^2\right)$$

How big is this?
Perfect Hashing - Space usage

**Step 1:** Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$

**Step 2:** The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$

How much space does this use? (Step 3)

- Immediately repeat if either
  a) $T$ has more than $n$ collisions
  b) some $T_i$ has a collision

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n_i^2)$

So the total space is...

$O(n) + \sum_i O(n_i^2) = O(n) + O\left(\sum_i n_i^2\right)$

Storing $h_i$ uses $O(1)$ space

How big is $\sum_i n_i^2$?

There are $\binom{n_i}{2}$ collisions in $T[i]$
Perfect Hashing - Space usage

Step 1: Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$.

Step 2: The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$.

How much space does this use?

(Step 3) Immediately repeat if either

a) $T$ has more than $n$ collisions

b) some $T_i$ has a collision

The size of $T$ is $O(n)$.

The size of $T_i$ is $O(n_i^2)$.

So the total space is...

$O(n) + \sum_i O(n_i^2) = O(n) + O\left(\sum_i n_i^2\right)$
Perfect Hashing - Space usage

**Step 1:** Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$

How big is $\sum_i n_i^2$?

There are $\binom{n_i}{2}$ collisions in $T[i]$ so there are $\sum_i \binom{n_i}{2}$ collisions in $T$

but we know that there are at most $n$ collisions in $T$ . . .

How much is the total space?

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n_i^2)$

Storing $h_i$ uses $O(1)$ space

So the total space is . . .

$$O(n) + \sum_i O(n_i^2) = O(n) + O\left( \sum_i n_i^2 \right)$$

how big is this?
Perfect Hashing - Space usage

**Step 1:** Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$

How much space does this use?

(Step 3)

Immediately repeat if either
a) $T$ has more than $n$ collisions
b) some $T_i$ has a collision

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n^{2i})$

So the total space is...

Storing $h_i$ uses $O(1)$ space

So the total space is...

$$O(n) + \sum_i O(n^{2i}) = O(n) + O\left(\sum_i n^{2i}\right)$$
Perfect Hashing - Space usage

**Step 1:** Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$

How big is $\sum_i n_i^2$?

There are $\binom{n_i}{2}$ collisions in $T[i]$ so there are $\sum_i \binom{n_i}{2}$ collisions in $T$

but we know that there are at most $n$ collisions in $T$...

$$\sum_i \frac{n_i^2}{4} \leq \sum_i \binom{n_i}{2} \leq n$$

Storing $h_i$ uses $O(1)$ space

So the total space is...

$$O(n) + \sum_i O(n_i^2) = O(n) + O\left(\sum_i n_i^2\right)$$
Perfect Hashing - Space usage

Step 1: Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$.

How much space does this use?

(Step 3) Immediately repeat if either

a) $T$ has more than $n$ collisions
b) some $T_i$ has a collision

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n^i)$

So the total space is...

$O(n) + \sum_i O(n^{2_i}) = O(n) + O\left(\sum_i n^{2_i}\right)$

Storing $h_i$ uses $O(1)$ space

How big is this?

There are $\binom{n_i}{2}$ collisions in $T[i]$ so there are $\sum_i \binom{n_i}{2}$ collisions in $T$

but we know that there are at most $n$ collisions in $T$...

\[
\sum_i \frac{n_i^2}{4} \leq \sum_i \binom{n_i}{2} \leq n
\]

\[
\binom{n_i}{2} = \frac{n_i(n_i-1)}{2} \geq \frac{n_i^2}{4}
\]
Perfect Hashing - Space usage

Step 1: Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$.

Step 2: The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n^{2i}$ using w.u hash function $h_i$.

How much space does this use?

(Step 3) Immediately repeat if either

a) $T$ has more than $n$ collisions
b) some $T_i$ has a collision

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n^{2i})$

So the total space is...

$O(n) + \sum_i O(n_i^{2i}) = O(n) + O\left(\sum_i n_i^{2i}\right)$

How big is $\sum_i n_i^{2i}$?

There are $\binom{n_i}{2}$ collisions in $T[i]$ so there are $\sum_i \binom{n_i}{2}$ collisions in $T$.

but we know that there are at most $n$ collisions in $T$...

$$\sum_i \frac{n_i^{2i}}{4} \leq \sum_i \binom{n_i}{2} \leq n$$

Storing $h_i$ uses $O(1)$ space.

how big is this?
**Perfect Hashing - Space usage**

**Step 1:** Insert everything into a hash table, $T$, of size $n$
using a weakly universal (w.u.) hash function, $h$

**Step 2:** The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$

How much space does this use?

Step 3) Immediately repeat if either

a) $T$ has more than $n$ collisions
b) some $T_i$ has a collision

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n_i^2)$

So the total space is...

$$O(n) + \sum_i O(n_i^2) = O(n) + O\left(\sum_i n_i^2\right)$$

Storing $h_i$ uses $O(1)$ space

How big is this?

There are $\binom{n_i}{2}$ collisions in $T[i]$ so there are $\sum_i \binom{n_i}{2}$ collisions in $T$

**but we know that there are at most $n$ collisions in $T$**

$$\sum_i n_i^2 \leq \sum_i \binom{n_i}{2} \leq n$$

or $$\sum_i n_i^2 \leq 4n$$
Perfect Hashing - Space usage

Step 1: Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$

How big is $\sum_i n_i^2$?

There are $\binom{n_i}{2}$ collisions in $T[i]$ so there are $\sum_i \binom{n_i}{2}$ collisions in $T$

but we know that there are at most $n$ collisions in $T$ . . .

$$\sum_i \frac{n_i^2}{4} \leq \sum_i \binom{n_i}{2} \leq n$$

or

$$\sum_i n_i^2 \leq 4n$$

Storing $h_i$ uses $O(1)$ space

So the total space is . . .

$$O(n) + \sum_i O(n_i^2) = O(n) + O \left( \sum_i n_i^2 \right) = O(n)$$
**Perfect Hashing - Space usage**

**Step 1:** Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$.

**Step 2:** The $n_i$ items in $T[i]$ are inserted into another hash table $T_i$ of size $n_i^2$ using w.u hash function $h_i$.

*(Step 3)* Immediately repeat if either

a) $T$ has more than $n$ collisions  
b) some $T_i$ has a collision

---

**How much space does this use?**

The size of $T$ is $O(n)$

The size of $T_i$ is $O(n_i^2)$

Storing $h_i$ uses $O(1)$ space.

*So the total space is…*

$$O(n) + \sum_i O(n_i^2) = O(n) + O \left( \sum_i n_i^2 \right) = O(n)$$
Perfect Hashing - Expected construction time

**Step 1:** Insert everything into a hash table, $T$, of size $n$ using a weakly universal (w.u.) hash function, $h$.

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using a weakly universal (w.u.) hash function, $h$

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(Step 3) Immediately repeat if either
a) $T$ has more than $n$ collisions
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The expected construction time for $T$ is $O(n)$

(we considered this on a previous slide)
Perfect Hashing - Expected construction time

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- we insert $n_i$ items into a table of size $m = n_i^2$
- then repeat if there was a collision
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The overall expected construction time is therefore:

$$
\mathbb{E}(\text{construction time}) = \mathbb{E}\left( \text{construction time of } T + \sum_i \text{construction time of } T_i \right)
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The expected construction time for $T$ is $O(n)$.

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The overall expected construction time is therefore:

$$E(\text{construction time}) = E \left( \text{construction time of } T + \sum_i \text{construction time of } T_i \right)$$

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**Perfect Hashing - Expected construction time**

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Perfect Hashing - Summary

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**Theorem**

The FKS hashing scheme:

- Has no collisions
- Every lookup takes $O(1)$ worst-case time,
- Uses $O(n)$ space,
- Can be built in $O(n)$ expected time.

*The look-up time is always $O(1)$*

1. Compute $i = h(x)$ ($x$ is the key)
2. Compute $j = h_i(x)$
3. The item is in $T_i[j]$
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