Hashing part one
Chaining, true randomness and universal hashing

Benjamin Sach
(based on slides by Markus Jalsenius)
Dictionaries

In a dictionary data structure we store \((key, value)\)-pairs such that for any key there is at most one pair \((key, value)\) in the dictionary.

Often we want to perform the following three operations:

- `add(x, v)` Add the pair \((x, v)\).
- `lookup(x)` Return \(v\) if \((x, v)\) is in dictionary, or \texttt{NULL}\ otherwise.
- `delete(x)` Remove pair \((x, v)\) (assuming \((x, v)\) is in dictionary).
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There are many data structures that will do this job, e.g.:

- Linked lists
- Binary search trees
- $(2,3,4)$-trees
- Red-black trees
- Skip lists
- van Emde Boas trees (later in this course)
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but none of them take \(O(1)\) worst case time for all operations... so *maybe* there is room for improvement?
Hash tables

We want to store $n$ elements from the universe, $U$, in a dictionary.

Typically $u = |U|$ is much, much larger than $n$. 

Universe $U$ containing $u$ keys.
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We write \( [m] \) to denote the set \( \{0, \ldots, m - 1\} \).
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We want to avoid collisions, i.e. $h(x) = h(y)$ for $x \neq y$. 

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Collisions can be resolved with chaining, i.e. linked list. 

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Time complexity

We cannot avoid collisions entirely since \( u \gg m \); some keys from the universe are bound to be mapped to the same position. (remember \( u \) is the size of the universe and \( m \) is the size of the table)

By building a hash table with chaining, we get the following time complexities:

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*So how long are these chains?*
True randomness

**Theorem**

Consider any \( n \) fixed inputs to the hash table (which has size \( m \)), i.e. any sequence of \( n \) add/lookup/delete operations.

Pick \( h \) uniformly at random from the set of all functions \( U \to [m] \).

The expected run-time per operation is \( O(1 + \frac{n}{m}) \), or simply \( O(1) \) if \( m \geq n \).
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Therefore, $\mathbb{E}(I_{x,y}) = \Pr(I_{x,y} = 1) = \Pr(h(x) = h(y)) = \frac{1}{m}$. 
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Let $x, y$ be two distinct keys from $U$. Let indicator r.v. $I_{x,y}$ be $1$ iff $h(x) = h(y)$, *iff* means *if and only if*.

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Observe that $N_x = \sum_{y \in T} I_{x,y}$ the keys in $T$
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Finally, we have that $E(N_x) = E \left( \sum_{y \in T} I_{x,y} \right) = \sum_{y \in T} E(I_{x,y}) = n \cdot \frac{1}{m} = \frac{n}{m}$ linearity of expectation.
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This has become rather cyclic... let’s try something else!
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As part of initialising the hash table,

we choose the hash function $h$ from $H$ randomly.
**Specifying the hash function**

**Problem:** how do we specify an *arbitrary* (e.g. a truly random) hash function?

For each key in $U$ we need to specify an arbitrary position in $T$,

this is a number in $[m]$, so requires $\log_2 m$ bits.

So in total we need $u \log_2 m$ bits, which is a ridiculous amount of space!

*(in particular, it's much bigger than the table :s)*

Instead, we define a set, or *family of hash functions*: $H = \{h_1, h_2, \ldots \}$.

As part of initialising the hash table,

we choose the hash function $h$ from $H$ randomly.

How should we specify the hash functions in $H$ and how do we pick one at random?
Weakly universal hashing

A set $H$ of hash functions is **weakly universal** if for any two distinct keys $x, y \in U$,

$$\Pr(h(x) = h(y)) \leq \frac{1}{m}$$

where $h$ is chosen uniformly at random from $H$. 
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**Observe**

The randomness here comes from the fact that $h$ is picked randomly.
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**Theorem**

Consider any $n$ fixed inputs to the hash table (*which has size* $m$),

i.e. any sequence of $n$ add/lookup/delete operations.

Pick $h$ uniformly at random from a weakly universal set $H$ of hash functions.

The expected run-time per operation is $O(1)$ if $m \geq n$. 
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Consider any $n$ fixed inputs to the hash table (which has size $m$), i.e. any sequence of $n$ add/lookup/delete operations.

Pick $h$ uniformly at random from a weakly universal set $H$ of hash functions.

The expected run-time per operation is $O(1)$ if $m \geq n$.

**Proof**

The proof we used for true randomness works here too (which is nice).
Constructing a weakly universal family of hash functions

- Suppose $U = [u]$, i.e. the keys in the universe are integers 0 to $u - 1$.
- Let $p$ be any prime bigger than $u$.
- For $a, b \in [p]$, let

$$h_{a,b}(x) = (ax + b \mod p) \mod m,$$

$$H_{p,m} = \{h_{a,b} | a \in \{1, \ldots, p - 1\}, b \in \{0, \ldots, p - 1\}\}.$$
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$H_{p,m}$ is a weakly universal set of hash functions.
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**Theorem**

$H_{p,m}$ is a weakly universal set of hash functions.

**Proof**


**Observe**

- $ax + b$ is a linear transformation which “spreads the keys” over $p$ values when taken modulo $p$. This does not cause any collisions.
- Only when taken modulo $m$ do we get collisions.
True randomness vs. weakly universal hashing

For both,

**true randomness**

\( h \) is picked uniformly from the set of all possible hash functions

and **weakly universal hashing**

\( h \) is picked uniformly from a weakly universal set of hash functions

we have seen that when \( m \geq n \),

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Since constructing a weakly universal set of hash functions seems much easier than obtaining true randomness, this is all good news!
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What about the length of the longest chain? (the longest linked list)
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Since constructing a weakly universal set of hash functions seems much easier than obtaining true randomness, this is all good news!

*isn’t it?*

What about the length of the *longest* chain? (the longest linked list)

If it is very long, some lookups could take a very long time…
If $h$ is selected uniformly at random from all functions $U ightarrow [m]$ then, over $m$ fixed inputs,

$$\Pr (\text{any chain has length } \geq 3 \log m ) \leq \frac{1}{m}. $$
Longest chain – true randomness

**Lemma**

If $h$ is selected uniformly at random from all functions $U \rightarrow [m]$ then, over $m$ fixed inputs,

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**Observe**

In this lemma we insert $m$ keys, i.e. $n = m$. 
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**Observe**

In this lemma we insert $m$ keys, i.e. $n = m$.

**Proof**

The problem is equivalent to showing that if we randomly throw $m$ balls into $m$ bins, the probability of having a bin with at least $3 \log m$ balls is at most $\frac{1}{m}$.
Let $X_1$ be the number of balls in the first bin.
Longest chain – true randomness

**Proof**

*continued…*

Let $X_1$ be the number of balls in the first bin.

Choose any $k$ of the $m$ balls *(we’ll pick $k$ in a bit)*
Proof (continued...)

Let $X_1$ be the number of balls in the first bin.

Choose any $k$ of the $m$ balls (we’ll pick $k$ in a bit)

the probability at all of these $k$ balls go into the first bin is $\frac{1}{m^k}$. 
Let $X_1$ be the number of balls in the first bin.

Choose any $k$ of the $m$ balls (we’ll pick $k$ in a bit)

the probability at all of these $k$ balls go into the first bin is $\frac{1}{m^k}$.

So, the union bound gives us

$$\Pr(X_1 \geq k) \leq \binom{m}{k} \cdot \frac{1}{m^k} \leq \frac{1}{k!}.$$
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Theorem

Let $V_1, \ldots, V_q$ be $q$ events. Then

$$\Pr\left( \bigcup_{i=1}^{q} V_i \right) \leq \sum_{i=1}^{q} \Pr(V_i).$$
Let $X_1$ be the number of balls in the first bin.

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$$\Pr(\text{at least one bin receives at least } k \text{ balls}) \leq m \cdot \Pr(X_1 \geq k) \leq \frac{m}{k!}.$$
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$$ \Pr(\text{at least one bin receives at least } k \text{ balls}) \leq m \cdot \Pr(X_1 \geq k) \leq \frac{m}{k!}. $$

*Now* we set $k = 3 \log m$ and observe that

$$ \frac{m}{k!} \leq \frac{1}{m} \text{ for } m \geq 2,$$

and we are done.
Let $X_1$ be the number of balls in the first bin.

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**Why is $\frac{m}{k!} \leq \frac{1}{m}$? (when $k = 3 \log m$)**

So, the $k$ terms:

\[
k! = k \times (k-1) \times (k-2) \ldots \times 2 \times 1
\]

\[
k! > 2 \times 2 \times 2 \ldots \times 2 \times 1 = 2^{k-1}
\]

Let $k = 3 \log m$...

\[
k! > 2^{(3 \log m - 1)} \geq 2^{2 \log m} = (2^{\log m})^2 = m^2
\]

By using the union bound again, we have that

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\Pr(\text{at least one bin receives at least } k \text{ balls}) \leq m \cdot \Pr(X_1 \geq k) \leq m \cdot \frac{1}{k!}
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so $m \cdot \frac{m}{k!} \leq \frac{m}{m^2} = \frac{1}{m}$

Now we set $k = 3 \log m$ and observe that $\frac{k}{m} \leq \frac{1}{m}$ for $m \geq 2$, and we are done.
Longest chain – true randomness

Lemma

If $h$ is selected uniformly at random from all functions $U \rightarrow [m]$ then, over $m$ fixed inputs,

$$\Pr(\text{any chain has length } \geq 3 \log m ) \leq \frac{1}{m}.$$ 

Observe

In this lemma we insert $m$ keys, i.e. $n = m$.

Proof

The problem is equivalent to showing that if we randomly throw $m$ balls into $m$ bins, the probability of having a bin with at least $3 \log m$ balls is at most $\frac{1}{m}$.
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The conclusion from previous slides is that with true randomness,
the longest chain is very short (at most $3 \log m$) with high probability.
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**Lemma**

If $h$ is picked uniformly at random from a weakly universal set of hash functions then, over $m$ fixed inputs,

$$\Pr \left( \text{any chain has length } \geq 1 + \sqrt{2m} \right) \leq \frac{1}{2}.$$
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$$\Pr \left( \text{any chain has length } \geq 1 + \sqrt{2m} \right) \leq \frac{1}{2}.$$ 

**Observe**

This rubbish upper bound of $\frac{1}{2}$ does not necessarily rule out the possibility that the tightest upper bound is indeed very small. However, the upper bound of $\frac{1}{2}$ is in fact tight!
Longest chain – weakly universal hashing

**Proof**

- For any two keys $x, y$, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$. 
Longest chain – weakly universal hashing

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- For any two keys $x, y$, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.
- Let r.v. $C$ be the total number of collisions: $C = \sum_{x,y \in T, x < y} I_{x,y}$. 
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- Let r.v. $C$ be the total number of collisions: $C = \sum_{x, y \in T, x < y} I_{x,y}$.
- Using linearity of expectation and $\mathbb{E}(I_{x,y}) = \frac{1}{m}$ ($h$ is weakly universal),

\[
\mathbb{E}(C) = \mathbb{E}\left( \sum_{x, y \in T, x < y} I_{x,y} \right) = \sum_{x, y \in T, x < y} \mathbb{E}(I_{x,y}) = \binom{m}{2} \cdot \frac{1}{m} \leq \frac{m}{2}.
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- by Markov’s inequality, $\Pr(C \geq m) \leq \frac{\mathbb{E}(C)}{m} \leq \frac{1}{2}$. 

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- Let r.v. $L$ be the length of the longest chain. Then $C \geq \binom{L}{2}$. 
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This is because a chain of length $L$ causes $\binom{L}{2}$ collisions!
**Longest chain – weakly universal hashing**

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- Let r.v. $L$ be the length of the longest chain. Then $C \geq \left( \frac{L}{2} \right)$.

- Now, $\Pr\left( \frac{(L-1)^2}{2} \geq m \right) \leq \Pr\left( \left( \frac{L}{2} \right) \geq m \right) \leq \Pr(C \geq m) \leq \frac{1}{2}$. 
For any two keys \( x, y \), let indicator r.v. \( I_{x,y} \) be 1 iff \( h(x) = h(y) \).

Let r.v. \( C \) be the total number of collisions: \( C = \sum_{x,y \in T, x < y} I_{x,y} \).

Using linearity of expectation and \( E(I_{x,y}) = \frac{1}{m} \) (\( h \) is weakly universal),

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E(C) = E\left( \sum_{x,y \in T, x < y} I_{x,y} \right) = \sum_{x,y \in T, x < y} E(I_{x,y}) = \binom{m}{2} \cdot \frac{1}{m} \leq \frac{m}{2}.
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by Markov’s inequality, \( \Pr(C \geq m) \leq \frac{E(C)}{m} \leq \frac{1}{2} \).

Let r.v. \( L \) be the length of the longest chain. Then \( C \geq \binom{L}{2} \).

Now, \( \Pr\left( \frac{(L-1)^2}{2} \geq m \right) \leq \Pr\left( \binom{L}{2} \geq m \right) \leq \Pr(C \geq m) \leq \frac{1}{2} \).

this is because \( \binom{L}{2} = \frac{L!}{2!(L-2)!} = \frac{L \cdot (L-1)}{2} \geq \frac{(L-1)^2}{2} \).
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- Now, $\Pr\left( \frac{(L-1)^2}{2} \geq m \right) \leq \Pr\left( \binom{L}{2} \geq m \right) \leq \Pr(C \geq m) \leq \frac{1}{2}$.
For any two keys \( x, y \), let indicator r.v. \( I_{x,y} \) be 1 iff \( h(x) = h(y) \).

Let r.v. \( C \) be the total number of collisions: \( C = \sum_{x, y \in T, x < y} I_{x,y} \).

Using linearity of expectation and \( \mathbb{E}(I_{x,y}) = \frac{1}{m} \) (\( h \) is weakly universal),

\[
\mathbb{E}(C) = \mathbb{E} \left( \sum_{x, y \in T, x < y} I_{x,y} \right) = \sum_{x, y \in T, x < y} \mathbb{E}(I_{x,y}) = \binom{m}{2} \cdot \frac{1}{m} \leq \frac{m}{2}.
\]

by Markov’s inequality, \( \Pr(C \geq m) \leq \frac{\mathbb{E}(C)}{m} \leq \frac{1}{2} \).

Let r.v. \( L \) be the length of the longest chain. Then \( C \geq \binom{L}{2} \).

Now, \( \Pr \left( \frac{(L-1)^2}{2} \geq m \right) \leq \Pr \left( \binom{L}{2} \geq m \right) \leq \Pr(C \geq m) \leq \frac{1}{2} \).
Longest chain – weakly universal hashing

**Proof**

- For any two keys $x, y$, let indicator r.v. $I_{x,y}$ be 1 iff $h(x) = h(y)$.
- Let r.v. $C$ be the total number of collisions: $C = \sum_{x, y \in T, x < y} I_{x,y}$.
- Using linearity of expectation and $\mathbb{E}(I_{x,y}) = \frac{1}{m}$ ($h$ is weakly universal),
  
  $$
  \mathbb{E}(C) = \mathbb{E}(\sum_{x, y \in T, x < y} I_{x,y}) = \sum_{x, y \in T, x < y} \mathbb{E}(I_{x,y}) = \left(\frac{m}{2}\right) \cdot \frac{1}{m} \leq \frac{m}{2}.
  $$

- by Markov’s inequality, $\Pr(C \geq m) \leq \frac{\mathbb{E}(C)}{m} \leq \frac{1}{2}$.
- Let r.v. $L$ be the length of the longest chain. Then $C \geq \left(\frac{L}{2}\right)$.
- Now, $\Pr\left(\frac{(L-1)^2}{2} \geq m\right) \leq \Pr\left(\left(\frac{L}{2}\right) \geq m\right) \leq \Pr\left(C \geq m\right) \leq \frac{1}{2}$.

By rearranging, we have that $\Pr\left(L \geq 1 + \sqrt{2m}\right) \leq \frac{1}{2}$, and we are done.
Conclusions

For both,

true randomness (\( h \) is picked uniformly from the set of all possible hash functions)

and weakly universal hashing

(\( h \) is picked uniformly from a weakly universal set of hash functions)

we have seen that when \( m \geq n \),

the expected lookup time in a hash table with chaining is \( O(1) \).

**Lemma**

If \( h \) is selected uniformly at random from all functions \( U \rightarrow [m] \) then,

\[
\Pr \left( \text{any chain has length } \geq 3 \log m \right) \leq \frac{1}{m}.
\]

**Lemma**

If \( h \) is picked uniformly at random from a weakly universal set of hash functions,

\[
\Pr \left( \text{any chain has length } \geq 1 + \sqrt{2m} \right) \leq \frac{1}{2}.
\]

(both Lemmas hold for \( m \) any fixed inputs)