Line Segment Intersections

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slides inspired by Marc van Kreveld
Introduction

Problem Given $n$ line segments, find all the intersections...
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A simple algorithm

One simple approach to this problem is to test every pair of line segments...

Let $s_i$ denote the $i$-th line intersection

For $i = 1, 2, \ldots, n$
For $j = 1, 2, \ldots, n$
If ($s_i$ intersects $s_j$) and ($i \neq j$)
output ($i, j$)
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Given two line segments $s_i$ and $s_j$ described by their end point coordinates
deciding whether (and where) they intersect
takes $O(1)$ time
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Why?
A simple algorithm

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For \( j = 1, 2, \ldots, n \)
If \( (s_i \text{ intersects } s_j) \) and \( (i \neq j) \)
output \((i, j)\)

Given two line segments \( s_i \) and \( s_j \) described by their end point coordinates

deciding whether (and where) they intersect

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Why?

Any computation on two objects with \( O(1) \) space descriptions takes \( O(1) \) time
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This algorithm runs in $O(n^2)$ time

*(because checking pair of lines takes $O(1)$ time)*
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(because checking pair of lines takes $O(1)$ time)

...can we do better?
If there are $n$ line segments... how many intersections can there be?
If there are $n$ line segments... how many intersections can there be?

Here there are 10 line segments and 25 intersections.
If there are \( n \) line segments…

how many intersections can there be?

Here there are 50 line segments

and 625 intersections
If there are $n$ line segments...

how many intersections can there be?

Here there are 50 line segments

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In general, there could be more than

$(\frac{n}{2})^2$ intersections
If there are \( n \) line segments... how many intersections can there be?

Here there are 50 line segments and 625 intersections.

In general, there could be more than \( \left( \frac{n}{2} \right)^2 \) intersections.

If we want to output all the intersections, we can’t possibly expect to do better than \( O(n^2) \) time in the worst-case.
Output sensitive algorithms

The time complexities of the algorithms we have seen so far (in this course) have only depended on the size on the input.
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Let $k$ denote the number of line segment intersections ($k$ is not provided in the input).
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The time complexity of the algorithm we will see in this lecture also depends on the size of the output.

Let $k$ denote the number of line segment intersections $(k$ is not provided in the input).

Here $n = 17$ and $k = 3$.
Output sensitive algorithms

The time complexities of the algorithms we have seen so far (in this course) have only depended on the size on the input.

The time complexity of the algorithm we will see in this lecture also depends on the size of the output.

Let $k$ denote the number of line segment intersections ($k$ is not provided in the input).

We will see an algorithm for line segment intersection which takes $O(n \log n + k \log n)$ time in the worst-case.

Here $n = 17$ and $k = 3$. 
Output sensitive algorithms

We will see an algorithm for line segment intersection which takes

\[ O(n \log n + k \log n) \] time in the worst-case

If \( k \) is small...

For example if \( k \leq 2n \) we get an

\[ O(n \log n) \] worst-case time

If \( k \) is big...

For example if \( k \geq \left( \frac{n}{2} \right)^2 \) we get an

\[ O(n^2 \log n) \] worst-case time

(which is worse than the simple algorithm)
Some simplifying restrictions

In the interest of simplicity, we don’t allow the input to contain any of the following:

- Horizontal line segments
- Overlapping line segments
- Two end points with the same y-coordinate
- Three (or more) line segments which intersect at the same point

All of these restrictions can be removed making the algorithm slightly more involved (without changing the time complexity)
A first observation
A first observation

$s_i$
A first observation

$s_i$

$y$-span of $s_i$
A first observation
A first observation

\[ s_i \]

\[ y\text{-span of } s_i \]

\[ s_j \]

\[ y\text{-span of } s_j \]
A first observation

If \( s_i \) and \( s_j \) don't have overlapping \( y \)-spans, they don't intersect.
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If $s_i$ and $s_j$ don’t have overlapping $y$-spans, they don’t intersect.

This suggests an overall approach to the problem...
A first observation

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This suggests an overall approach to the problem...

sweep a horizontal line through the plane from top to bottom

finding intersections as we go
A first observation

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*sweep a horizontal line through the plane from top to bottom* finding intersections as we go.
A first observation

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If $s_i$ and $s_j$ don’t have overlapping $y$-spans, they don’t intersect.

This suggests an overall approach to the problem...

*sweep a horizontal line through the plane from top to bottom* finding intersections as we go.
Adjacent line segments and a second observation

These two line segments are adjacent at this $y$-coordinate.
Adjacent line segments and a second observation

These two line segments are adjacent at this $y$-coordinate

(there is no line segment between them)
Adjacent line segments and a second observation
Adjacent line segments and a second observation

These two line segments
Adjacent line segments and a second observation

These two line segments are adjacent at this $y$-coordinate.
Adjacent line segments and a second observation

These two line segments are \textit{adjacent} at this $y$-coordinate

but they aren’t \textit{adjacent} at this $y$-coordinate
Adjacent line segments and a second observation
Adjacent line segments and a second observation

These two line segments
Adjacent line segments and a second observation

These two line segments never become *adjacent*.
Adjacent line segments and a second observation

These two line segments never become *adjacent* so they can’t intersect.
Adjacent line segments and a second observation

These two line segments never become *adjacent* so they can’t intersect.

Two line segments $s_i$ and $s_j$ which are never adjacent don’t intersect.
The overall approach

The overall approach is to imagine a horizontal line passing through the plane from the top to the bottom. This is called a plane sweep.
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We cannot hope to process the sweep line at every \( y \)-coordinate so the sweep line jumps between interesting positions called event points.
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The event points are the end points of line segments and the intersection points.
The overall approach

The overall approach is to imagine a horizontal line passing through the plane from the top to the bottom, which is called a plane sweep.

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The *event points* are

the end points of line segments

and the *intersection points*
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The event points are the end points of line segments and the intersection points. We will have to detect the intersection points on the fly (before we get to them).
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*This is called a plane sweep.*

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The *event points* are the end points of line segments and the *intersection points*.

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The event points are the end points of line segments and the intersection points.

We will have to detect the intersection points on the fly (before we get to them).

The number of event points is $O(n + k)$. 

---

$O(n + k)$
The status of the sweep line

The status of the sweep line

is the set of line segments which currently intersect the sweep line

ordered from left to right by where they intersect

(i.e. in the order given by the ■ )
The status of the sweep line

The *status* of the sweep line
is the set of line segments which currently intersect the sweep line
ordered from left to right by where they intersect
(i.e. in the order given by the ■ )

this is first

this is last
The status of the sweep line

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(i.e. in the order given by the ■ )
The status of the sweep line is the set of line segments which currently intersect the sweep line ordered from left to right by where they intersect (i.e. in the order given by the □).

Fact: the status of the sweep line can only change at event points (i.e. at an end point or an intersection).
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Fact the status of the sweep line can only change at event points (i.e. at an end point or an intersection)

the □ move but the order stays the same
The status of the sweep line

The status of the sweep line is the set of line segments which currently intersect the sweep line ordered from left to right by where they intersect (i.e. in the order given by the direction of the moves).

Fact the status of the sweep line can only change at event points (i.e. at an end point or an intersection).

The status moves but the order stays the same.
The status of the sweep line

The *status* of the sweep line is the set of line segments which currently intersect the sweep line ordered from left to right by where they intersect (i.e. in the order given by the ■).

**Fact** the status of the sweep line can only change at event points (*i.e.* at an end point or an intersection)

*■ move but the order stays the same*

We will store the status of the sweep line in a data structure which allows efficient updates (*more details later*)
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The status of the sweep line tells us which line segments are currently adjacent
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*Why is this useful?*
The status of the sweep line

The *status* of the sweep line is the set of line segments which currently intersect the sweep line ordered from left to right by where they intersect (i.e. in the order given by the ■).

The status of the sweep line tells us which line segments are currently adjacent.

*Why is this useful?*

Two line segments which *never* become adjacent cannot intersect.
The status of the sweep line

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Why is this useful?

The status of the sweep line tells us which line segments are currently adjacent

*Why is this useful?*

two line segments which *never* become adjacent cannot intersect

We will detect each upcoming intersection when the corresponding line segments first become adjacent
The status of the sweep line

The *status* of the sweep line is the set of line segments which currently intersect the sweep line ordered from left to right by where they intersect (i.e. in the order given by the ).

![Diagram of line segments]

**Why is this useful?**

Two line segments which *never* become adjacent cannot intersect.

We will detect each upcoming intersection when the corresponding line segments first become adjacent.
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The status of the sweep line tells us which line segments are currently adjacent.

Why is this useful?

two line segments which *never* become adjacent cannot intersect

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Updating the sweep line

Every time the sweep line moves to the next event point, we update the status data structure.
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If the event point is the top of a line segment, we insert it into the status data structure (at the appropriate place).
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Every time the sweep line moves to the next event point, we update the status data structure.

If the event point is the top of a line segment, we insert it into the status data structure (at the appropriate place).

We then check whether this segment will intersect either of the adjacent segments.
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Every time the sweep line moves to the next event point, we update the status data structure.

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Updating the sweep line

Every time the sweep line moves to the next event point, we update the status data structure.

If the event point is the top of a line segment, we insert it into the status data structure (at the appropriate place).

We then check whether this segment will intersect either of the adjacent segments.
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Will these intersect?

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the bottom of a line segment, we **delete** it from the status data structure
an intersection point, we **swap** the two line segments in the status data structure

*and we always check whether we have discovered any new event points (specifically intersections)*
How do we keep track of the event points?

At the start of the algorithm, we are aware of $2n$ event points, one for each end of each line segment.
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We keep track of the event points using a Priority Queue. Every event point is **INSERTED** as it is discovered (with its $y$ value as the key). We can then use **DELETEMIN** to recover the next event point.
Can we miss out on an intersection?

If $s_i$ and $s_j$ intersect

they must become adjacent at some $y$-coordinate

(before they intersect)
Can we miss out on an intersection?

If \( s_i \) and \( s_j \) intersect

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In particular, they must become adjacent at some event point

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Can we miss out on an intersection?

If $s_i$ and $s_j$ intersect, they must become adjacent at some $y$-coordinate (before they intersect).

In particular, they must become adjacent at some *event point* with a higher $y$-coordinate.

This is because the status of the sweep line doesn’t change between event points.
Can we miss out on an intersection?

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they must become adjacent at some $y$-coordinate

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In particular, they must become adjacent at some **event point**

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This is because the status of the sweep line doesn’t change between event points

The formal proof then follows by induction.
Can we find the same event point twice?

Consider the line segments shown...
Can we find the same event point twice?

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Can we find the same event point twice?

Consider the line segments shown...
Can we find the same event point twice?

Consider the line segments shown...

when we process this event point,

we discover this event point
Can we find the same event point twice?

Consider the line segments shown... when we process this event point,

we rediscover this event point
Can we find the same event point twice?

Consider the line segments shown... when we process this event point, we rediscover this event point again.
Can we find the same event point twice?

Consider the line segments shown...
Can we find the same event point twice?
Can we find the same event point twice?

That is, we *can* discover the same event point more than once.
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There are (at least) two ways to deal with this:

1. Check whether we already found the new point
   - by looking it up in the priority queue

2. Don’t worry about it *(INSERT it anyway)*
   but make sure that when we process an event point, we only process it once
   - by checking that the priority queue didn’t return the same event point as last time
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There are (at least) two ways to deal with this:

1. Check whether we already found the new point - by looking it up in the priority queue

2. Don’t worry about it (INSERT it anyway) but make sure that when we process an event point, we only process it once - by checking that the priority queue didn’t return the same event point as last time

either approach gives the same time complexity
How do we implement the status data structure?

We need a data structure to store the *status* of the sweep line

i.e. the set of line segments which currently intersect the sweep line

ordered from left to right by where they intersect

(i.e. in the order given by the ■)
How do we implement the status data structure?

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The operations we need are

**INSERT, DELETE, FIND** and **Predecessor/Successor**
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![Diagram of line segments with arrows indicating insertion and deletion]

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we also need **SWAP** but this can be done using

**INSERT** and **DELETE**
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*however*, we need to be careful about the *keys*
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the description of $s_i$ (i.e. its end points) as the key
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Actually, all we require is that the keys have an order

and we can compare two keys in $O(1)$ time
Time Complexity (sketch)

The algorithm moves the sweep line $O(n + k)$ times, once for each event point.
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If the status data structure and priority queue structures

are implemented so that their operations take $O(\log n)$ time

(e.g. with a self-balancing tree and a binary heap, respectively)
Time Complexity (sketch)

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   once for each event point.

If the status data structure and priority queue structures 
   are implemented so that their operations take \( O(\log n) \) time 
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The overall complexity then becomes \( O(n \log n + k \log n) \) 
   as claimed.
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If the status data structure and priority queue structures are implemented so that their operations take $O(\log n)$ time (e.g. with a self-balancing tree and a binary heap, respectively)

The overall complexity then becomes $O(n \log n + k \log n)$ as claimed.

This is because we do a $O(n + k)$ operations on each data structure while moving the sweep line.
Summary

We have seen an algorithm for line segment intersection which runs in $O(n \log n + k \log n)$ time where $n$ is the number of line segments and $k$ is the number of intersections.

The approach is to move a horizontal line through the plane which jumps between all the interesting positions. The efficiency relies on using a Self-balancing search tree and a Binary Heap.

We put quite a few restrictions on the input, fixing these is fiddly but not difficult.

In the original paper, they suggest adding random noise to the points to avoid the restrictions.
Dealing with the restrictions

In the interest of simplicity, we didn’t allow the input to contain any of the following:

- Horizontal line segments
- Overlapping line segments
- Two end points with the same y-coordinate
- Three or more line segments which intersect at the same point

All of these restrictions can be removed making the algorithm slightly more involved (hints are given for the interested)