

Descent into Cache-Oblivion

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March 13, 2008

Outline

What is an External Memory Algorithm?

- The External Memory Model

- Two (simple) examples

Searching Cache-Obliviously

- Why aren't classic Binary Search Trees good enough?

- Static Cache-Oblivious Search Trees

- Making things Dynamic - allowing insertions and deletions

Sorting Cache-Obliviously

- Why isn't (binary) MergeSort good enough?

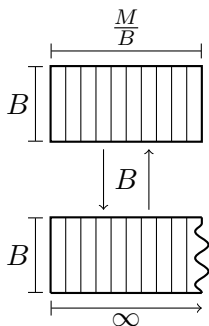
- FunnelSort - a Cache-Oblivious MergeSort

Summary and some Empirical Results

External Memory (EM) algorithms

The Problem

- ▶ Analysis in the RAM model relies on constant time memory access.

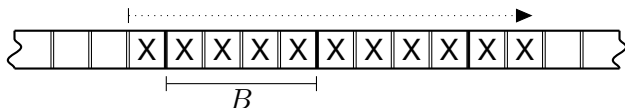


The Solution: The External Memory Model

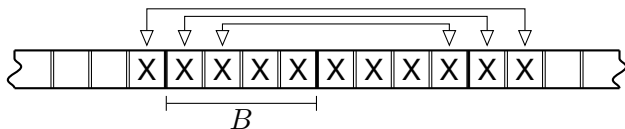
- ▶ Two levels of memory, internal (of size M) and external (unbounded).
- ▶ Data transfer occurs in blocks of size B
- ▶ We analyse asymptotic I/O complexity.
- ▶ A *Cache-Aware* algorithm knows M and B
- ▶ A *Cache-Oblivious* algorithm doesn't

Two (simple) examples of Cache-Oblivious algorithms

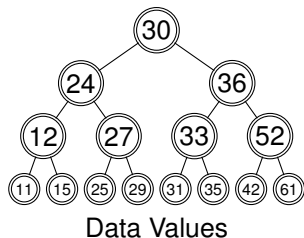
- ▶ **Example 1:** *Scanning* requires $\lceil \frac{N}{B} \rceil + 1 \in \Theta(\frac{N}{B})$ I/Os



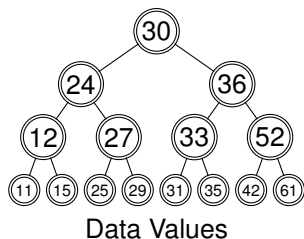
- ▶ **Example 2:** *Array Reversal* requires $\lceil \frac{N}{B} \rceil + 1 \in \Theta(\frac{N}{B})$ I/Os



Why aren't classic Binary Search Trees good enough?

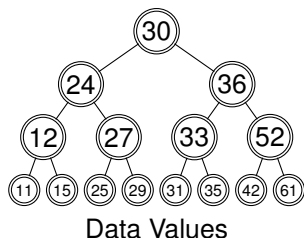


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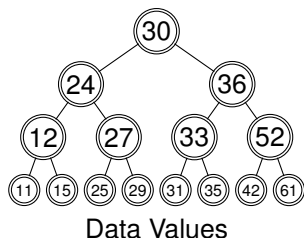
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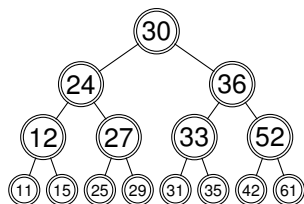
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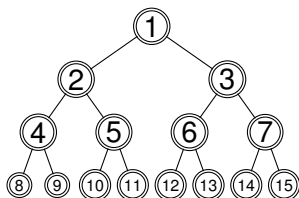


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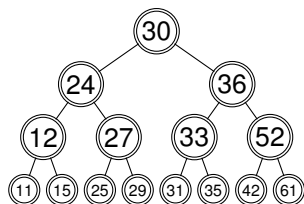
Data Values



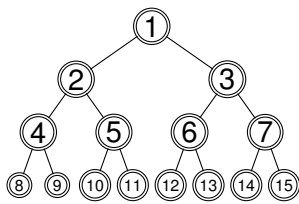
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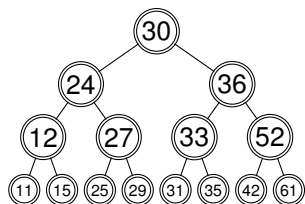
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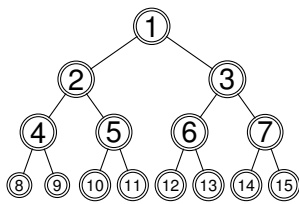
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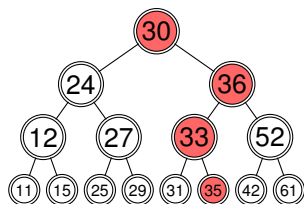
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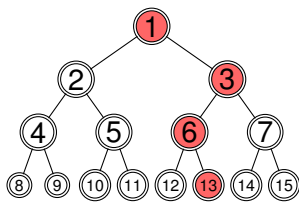
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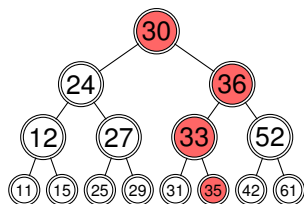
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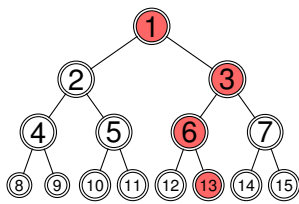
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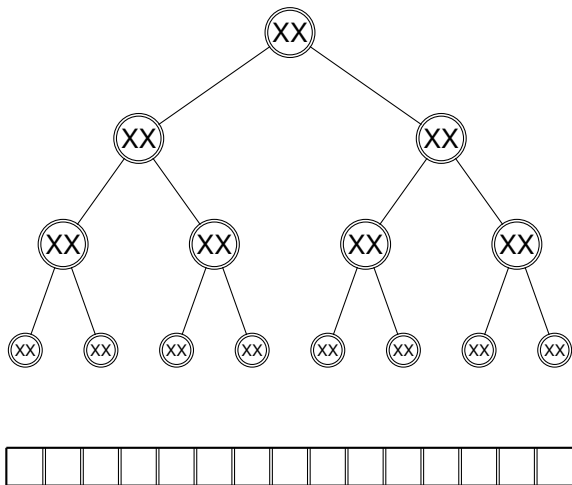


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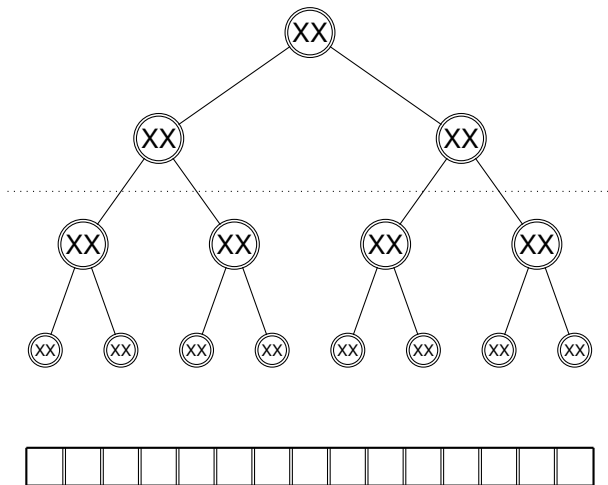
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$$O(\log N - \log B) = O(\log(N/B)) \text{ I/Os}$$

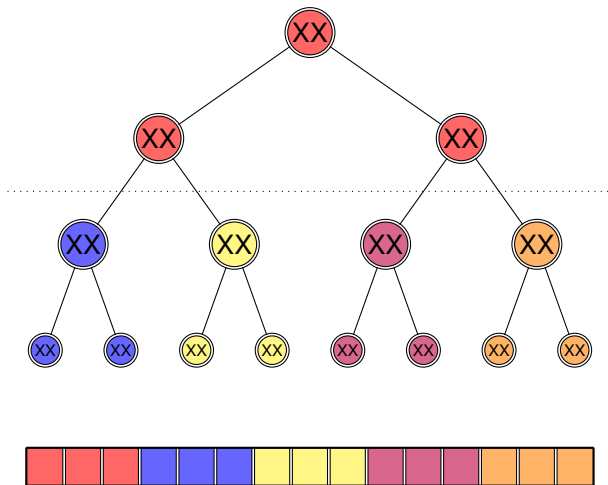
An example of the Van Emde Boas layout



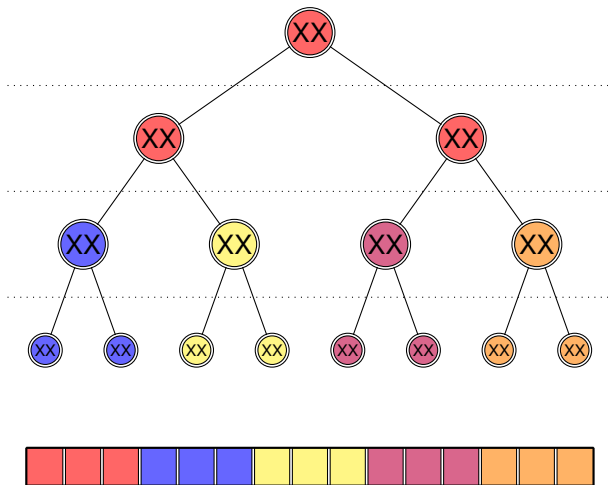
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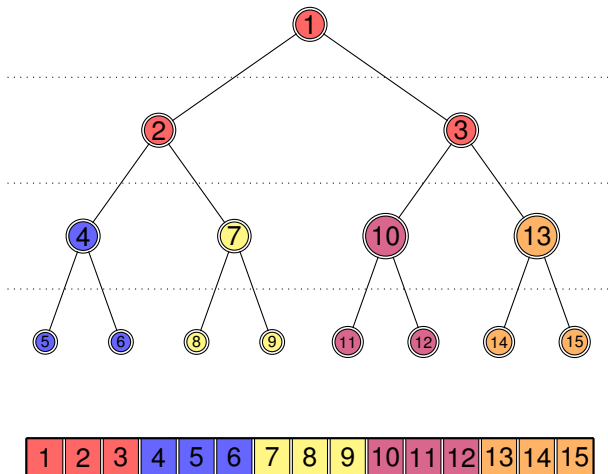
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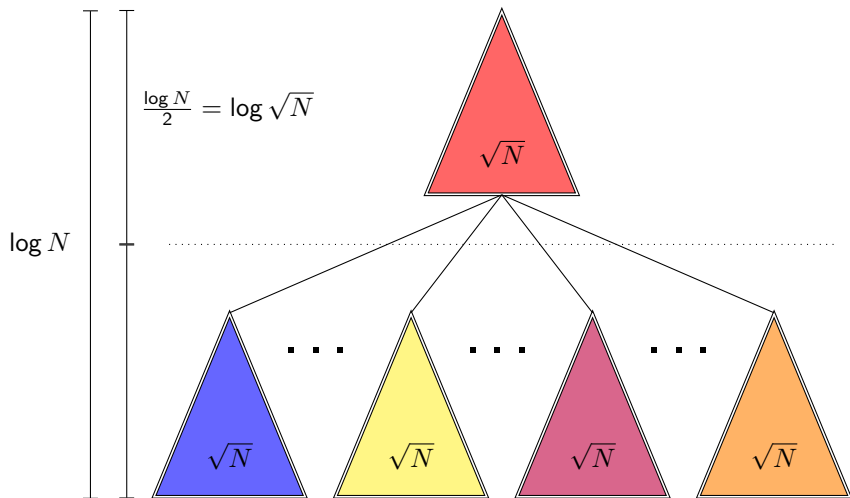
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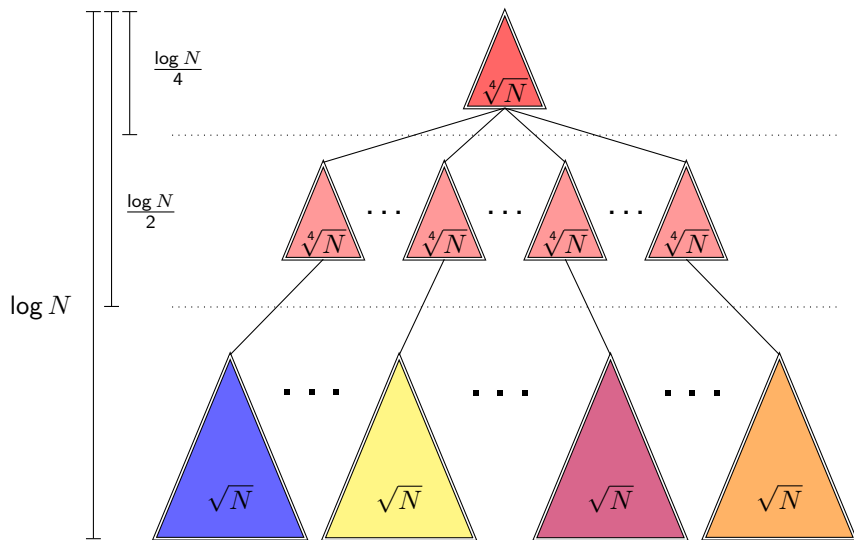
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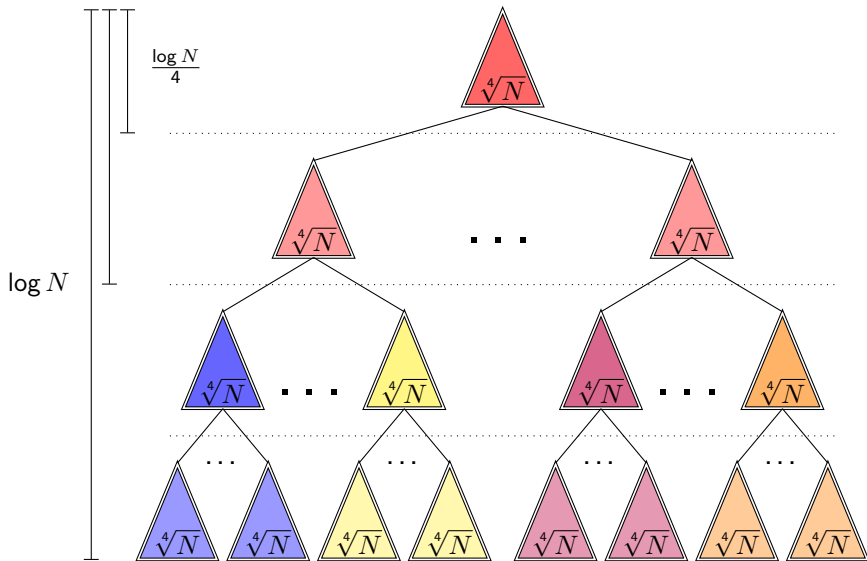
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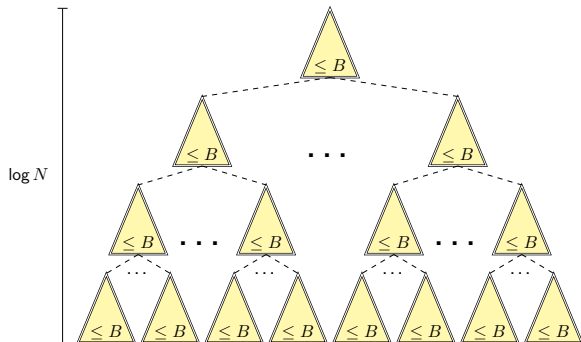
The Van Emde Boas layout -top expanded



The Van Emde Boas layout - level of detail $\sqrt[4]{N}$

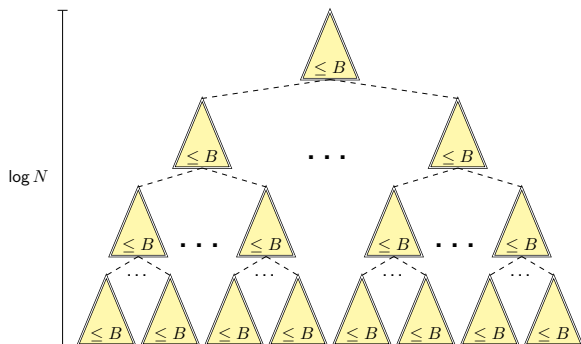


The Van Emde Boas layout - level of detail B



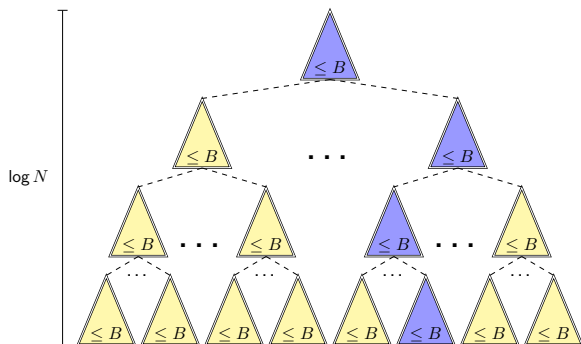
The Van Emde Boas layout - level of detail B

- ▶ conceptually recurse until all subtrees are smaller than B
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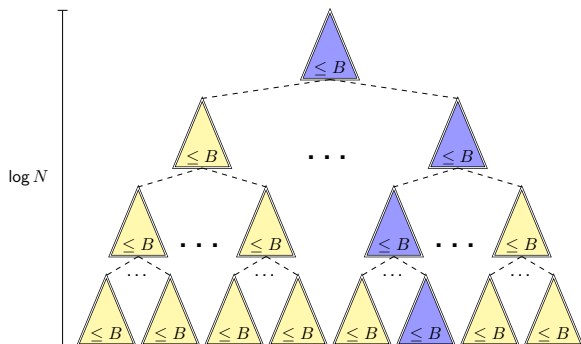
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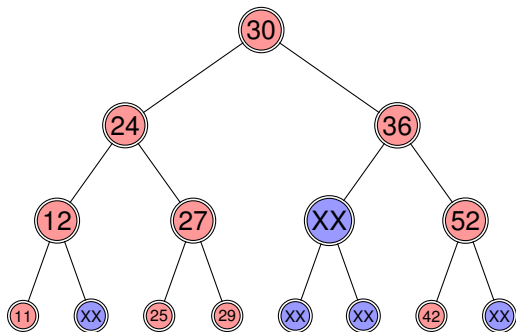
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$$\sim \log N / \log \sqrt{B} \in O(\log N / \log B) = O(\log_B N)$$

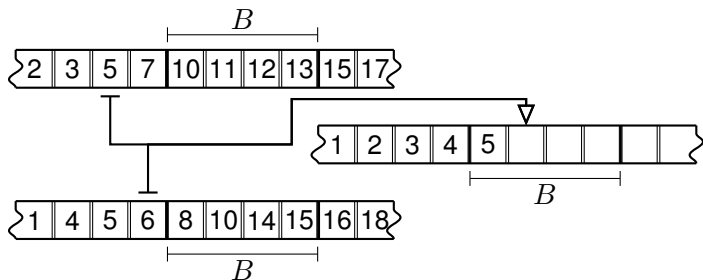


Making things Dynamic - allowing insertions and deletions

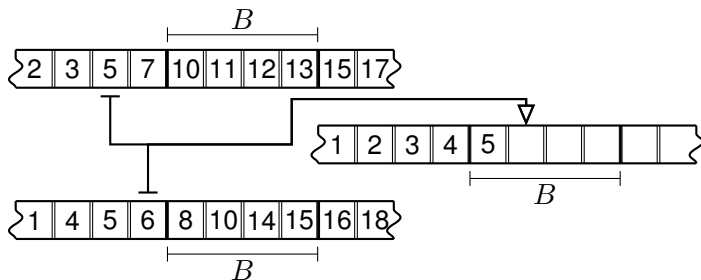


- ▶ Idea : embed the dynamic tree in a larger static tree
- ▶ If the dynamic tree becomes too unbalanced, re-distribute nodes
- ▶ If the dynamic tree becomes too large, re-construct
- ▶ Insertions and Deletions can be performed in $O(\log^2 N/B)$ I/Os

Why isn't (binary) MergeSort good enough?

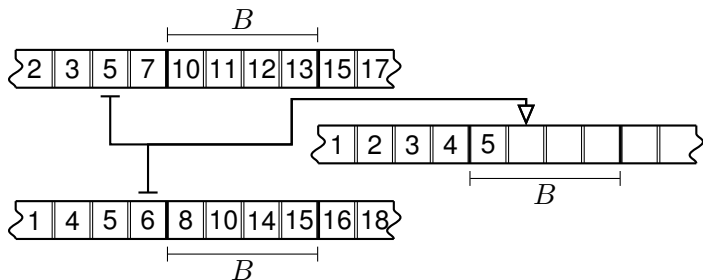


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$$T(N) = 2T(N/2) + O(N/B)$$

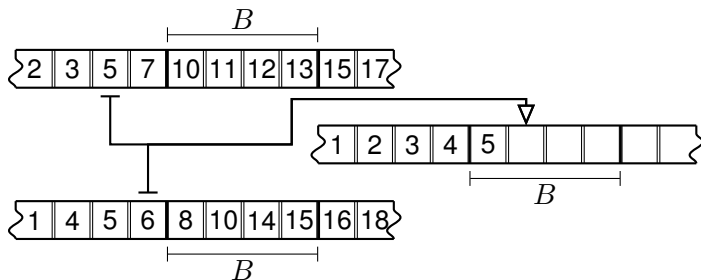
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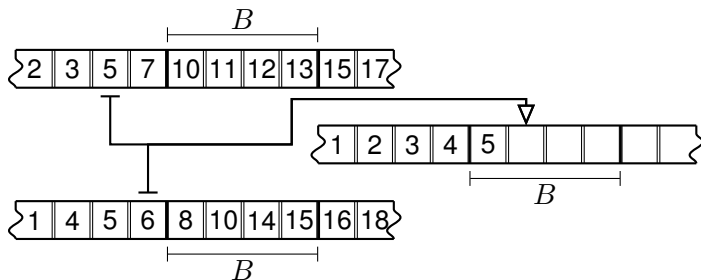
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(M/B) -way MergeSort gives $\Theta(\frac{N}{B} \log_{\frac{M}{B}} \frac{N}{B})$ I/Os (Cache-Aware)

A Cache-Oblivious MergeSort?

What if we had a black box which merged K sorted lists of total size K^3 in $O(\frac{K^3}{B} \log_{\frac{M}{b}} \frac{K^3}{B} + K)$ I/Os?

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1. Split the array into $K = N^{1/3}$ segments of length $N/K = N^{2/3}$
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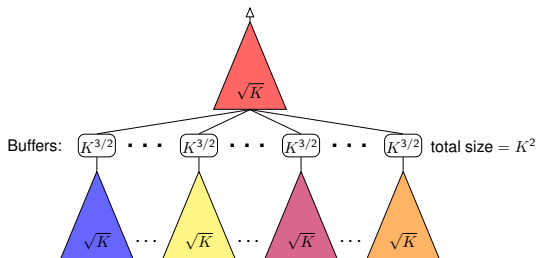
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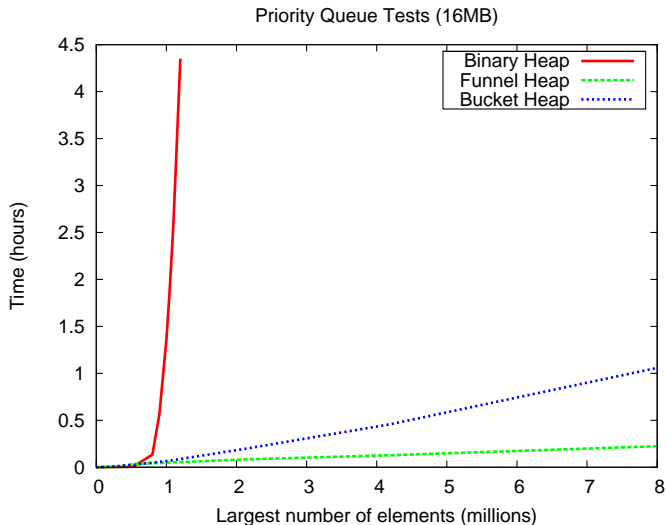
(assuming the tall cache assumption that $M > B^2$)

A K-funnel - the idea

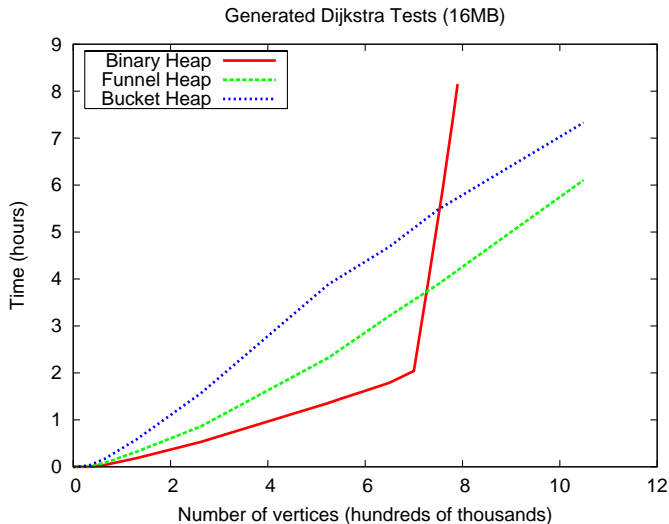


- ▶ Look at the largest level of detail J such that a J -funnel occupies less than $M/4$ space
- ▶ We can hold a J -funnel and one block of each of its input buffers in memory.
- ▶ Empty input buffers are completely refilled. This may push the original Funnel out of memory, which the 'new' J^3 elements pay for.

Some empirical results (1)



Some empirical results (2)



Summary and Questions

- ▶ Scanning/Array Reversal: $\Theta(N/B)$ I/Os
- ▶ Searching: $\Theta(\log_B N)$ I/Os
- ▶ Sorting: $\Theta(\frac{N}{B} \log_{\frac{M}{B}} \frac{N}{B})$ I/Os

Questions?