Parameterized Matching in the Streaming Model

Markus Jalsenius[1], Benny Porat[2] and *Benjamin Sach*[3]

(1) University of Bristol, UK
(2) Bar-Ilan University, Israel
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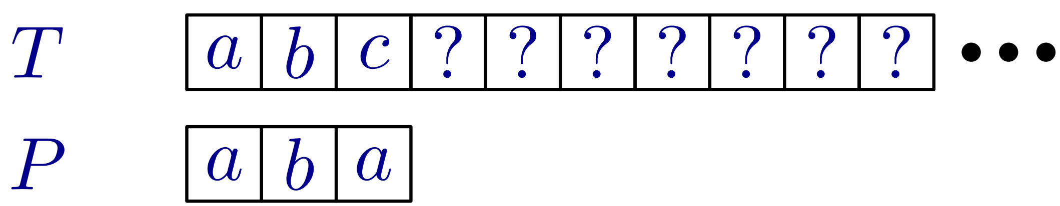
- We assume we have P in advance but T arrives online. . .



Goal: Decide whether P **matches** the last $|P|$ text characters. . .
before the next character arrives

- The definition of a **match** depends on the problem

- We care about **worst-case** time per text character
and using as little space as possible

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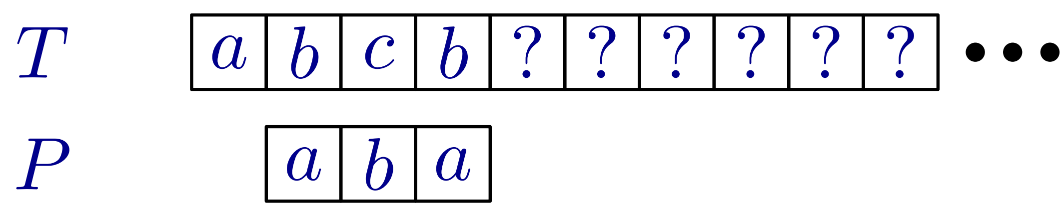

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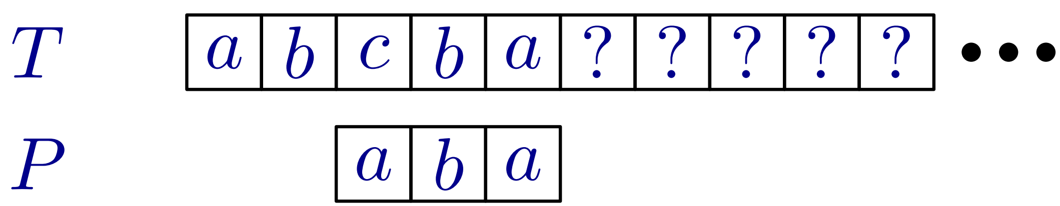

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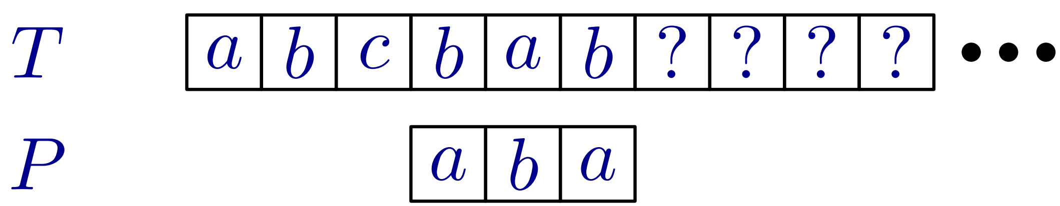

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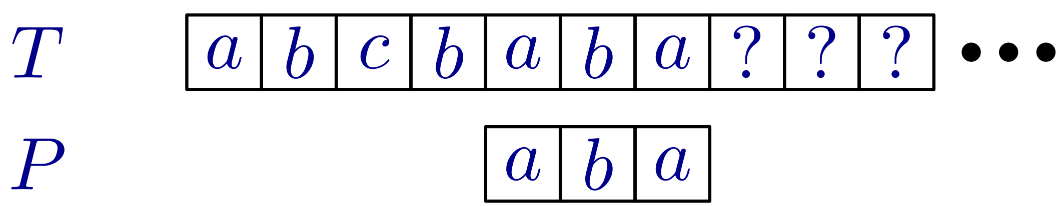

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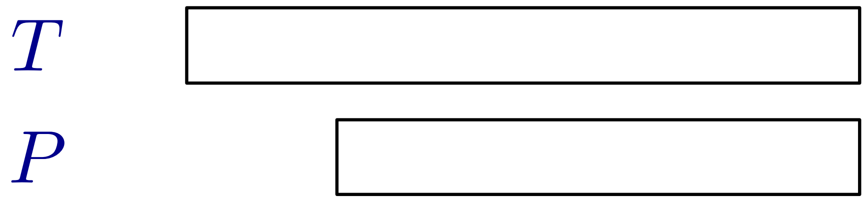
Streaming pattern matching in small space

What's the least space we can get away with?

- *Surely*, we have to store the pattern?



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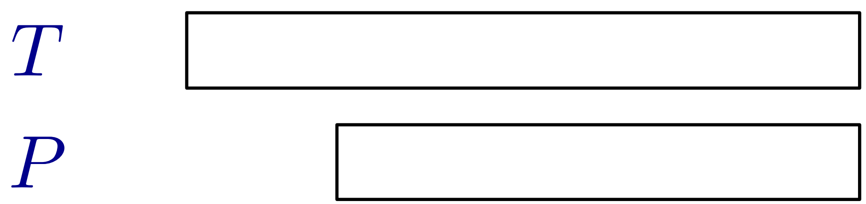

Porat, Porat FOCS'09

- Exact pattern matching can be solved in $O(\log |P|)$ space and $O(\log |P|)$ time per character

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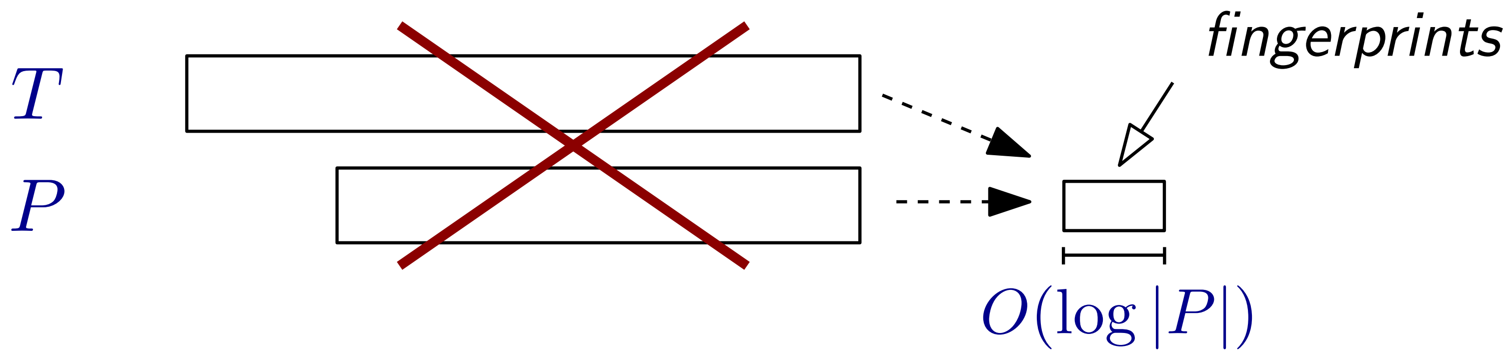

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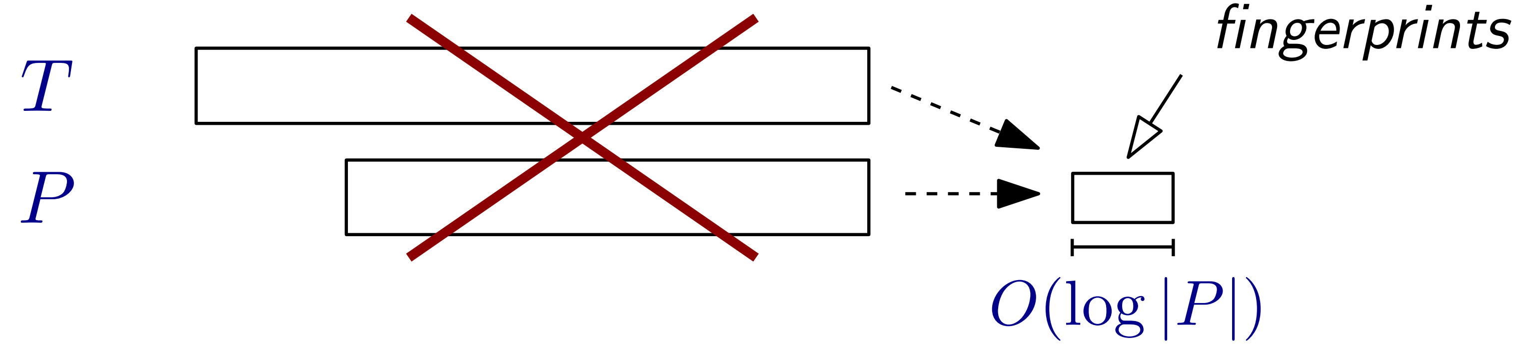

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The algorithm is randomised,

- it might make mistakes but it's correct with high probability

(at least $1 - 1/|T|^3$)

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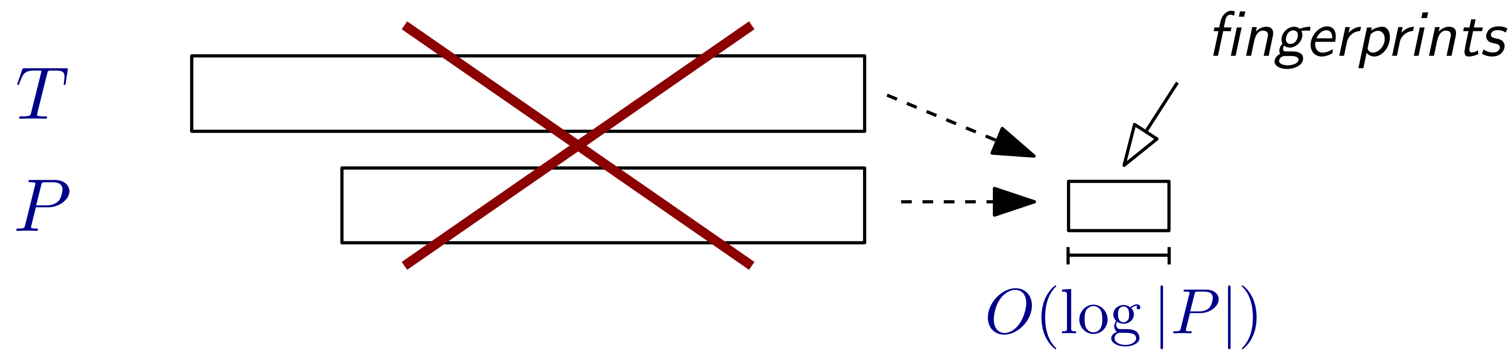

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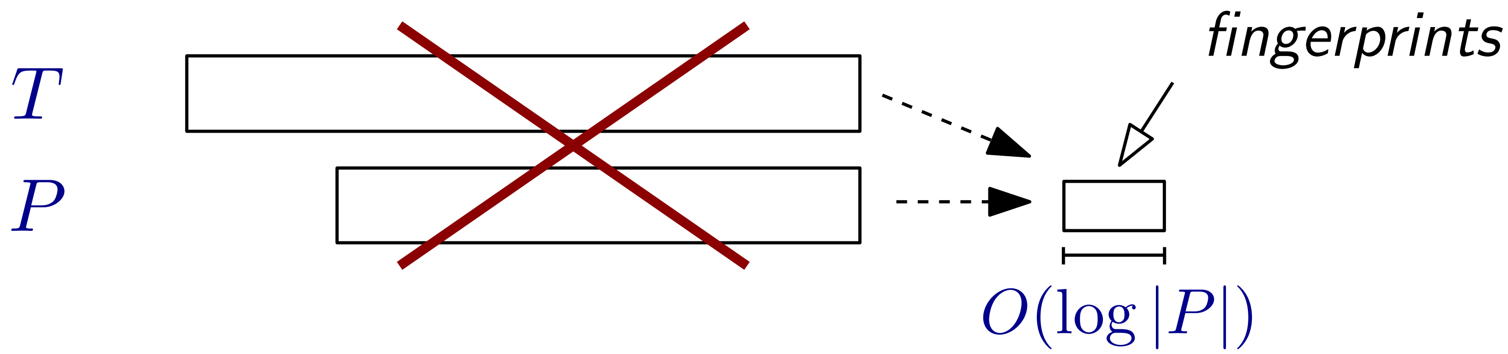

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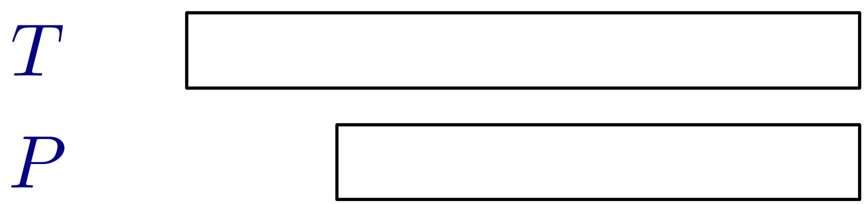

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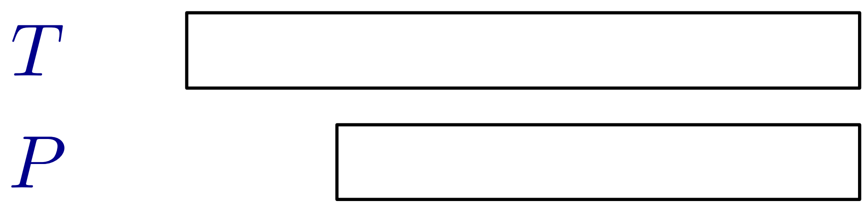

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- Exact pattern matching can be solved in
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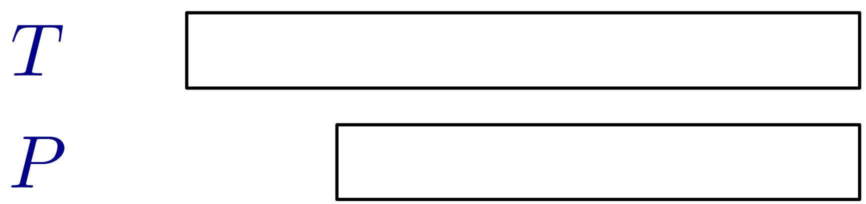

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  (this algorithm is also randomised)

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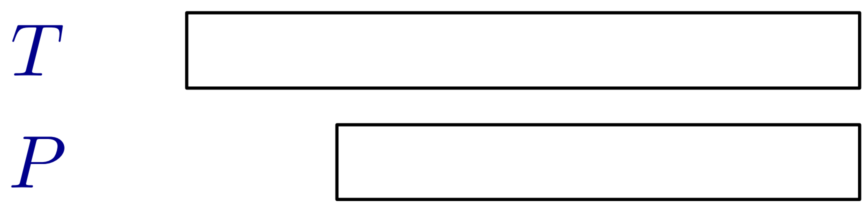

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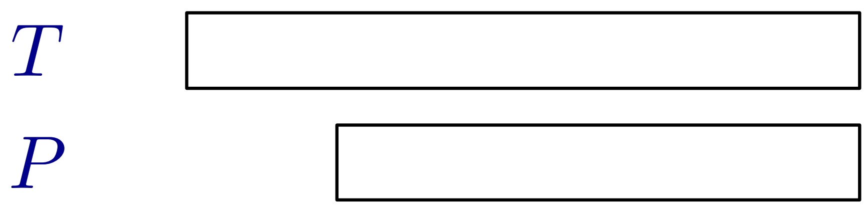

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What else can be solved in small space?

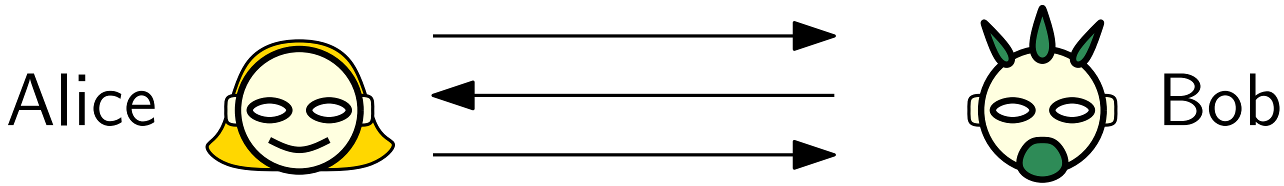
Streaming pattern matching in small space

Clifford, Jalsenius, Porat, **S.** *CPM'11*

We showed randomised space lower bounds of $\Omega(|P|)$ bits for:

Hamming distance, *Exact matching with wildcards*, L_1, L_2 *and* L_∞ *distances*, *Edit distance and Swap matching*

as well as any algorithm performing convolutions

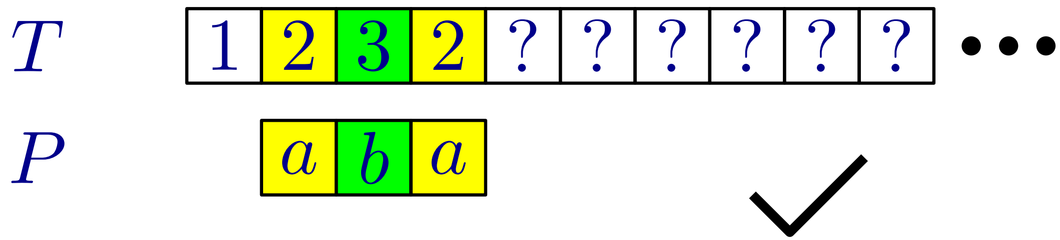


We did this by showing that better algorithms would give impossibly good communication protocols

Parameterized matching in a stream

- In parameterized matching (p-matching),

  the alphabet can be relabelled



- The alphabet mapping can differ for each pattern/text alignment
- The mapping must be one-to-one (injective)

P p-matches $T[i, i + |P| - 1]$ iff there is
a one-to-one f s.t. $f(P[j]) = T[i + j]$ for all j

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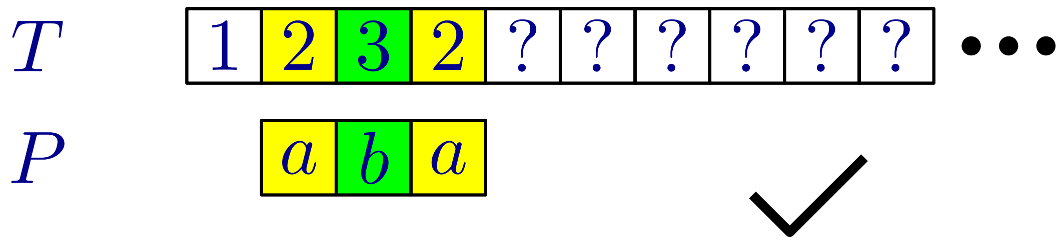

$a \to 2$, $b \to 3$ gives a mapping

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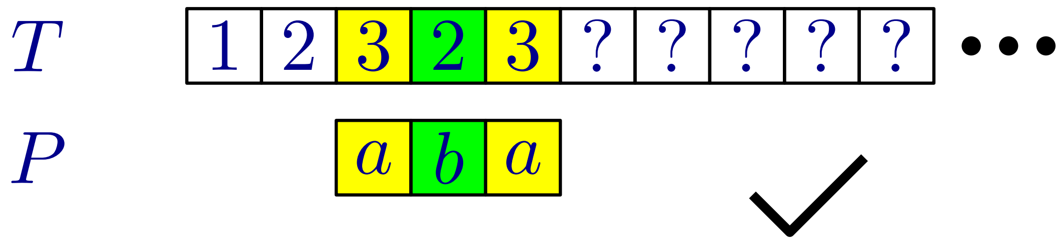

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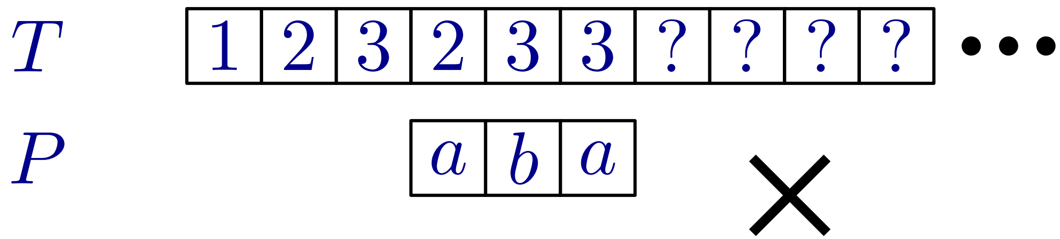

there is no mapping - a can't map to 2 *and* 3

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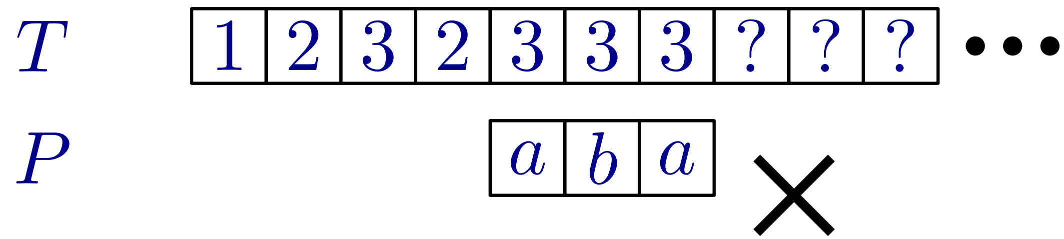

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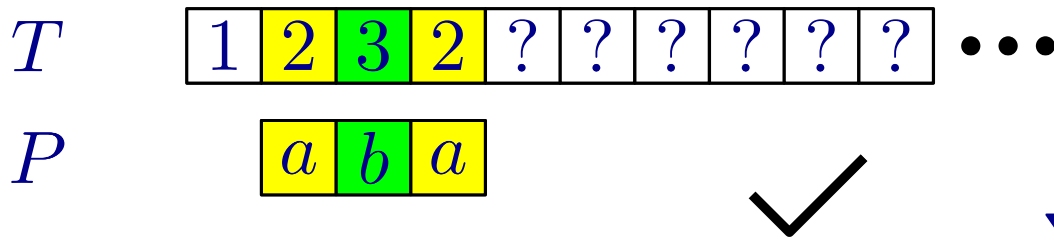

Σ is the alphabet

Our Results

Parameterized matching can be solved in $O(|\Sigma| \log |P|)$ space and:

- $O(1)$ time per character when $|\Sigma| = \{1, 2, 3, \ldots, |\Sigma|\}$
- $O(\sqrt{\log |\Sigma| / \log \log |\Sigma|})$ time per character for general Σ

Parameterized matching in a stream

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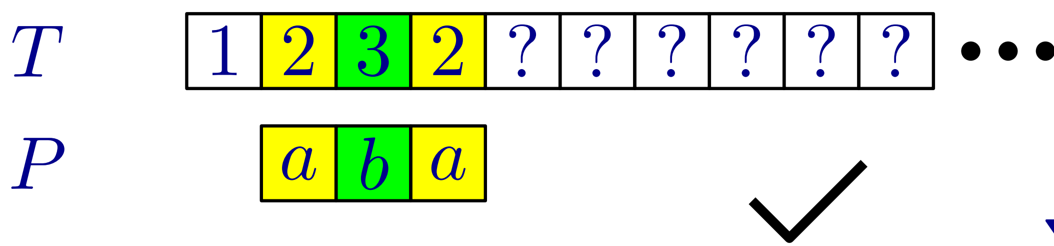

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Both algorithms are randomised (Monte-Carlo)

We also give an $\Omega(|\Sigma|)$ bit randomised space lower bound

Parameterized matching in a stream

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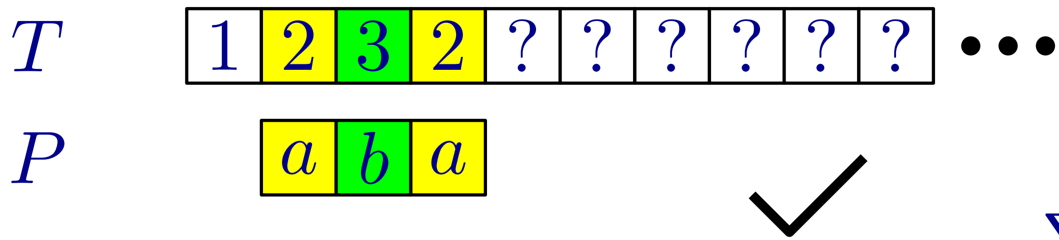

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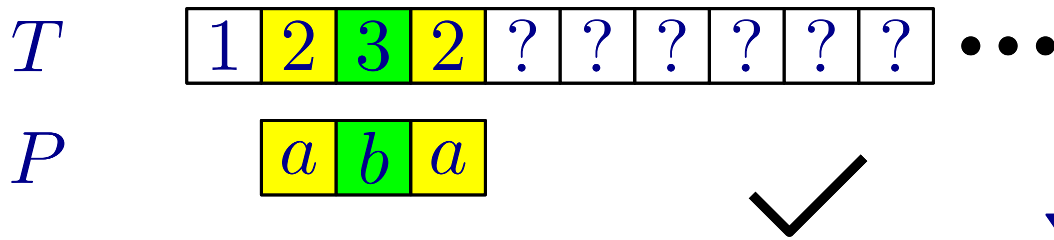

Σ is the alphabet

Our Results

Parameterized matching can be solved in $O(|\Pi| \log |P|)$ space and:

- $O(\sqrt{\log |\Pi| / \log \log |\Pi|})$ time per character for general Π

where $\Pi \subseteq \Sigma$ is the set of symbols in P
which are *allowed* to be relabelled

Parameterized matching in a stream

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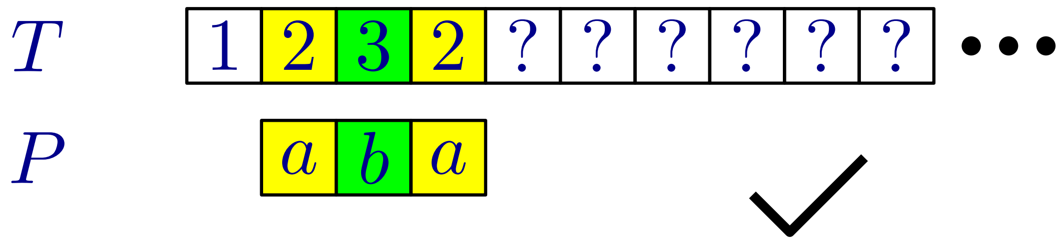

Parameterized matching can be solved in $O(|\Sigma| \log |P|)$ space and:

- $O(1)$ time per character when $|\Sigma| = \{1, 2, 3, \ldots, |\Sigma|\}$

The remainder of the talk will focus on this result
(the others are simple generalisations)

Predecessor Strings

S_1	a	b	a	c	c	b	a	b	c	b
S_2	b	c	b	a	a	c	b	c	a	c

Predecessor Strings

S_1	a	b	a	c	c	b	a	b	c	b
S_2	b	c	b	a	a	c	b	c	a	c

S_1 p-matches S_2 with mapping $a \to b$, $b \to c$, $c \to a$ (from S_1 to S_2)

Predecessor Strings

pred(S_1)	0	0	2	0	1	4	4	2	4	2

S_1	a	b	a	c	c	b	a	b	c	b
S_2	b	c	b	a	a	c	b	c	a	c

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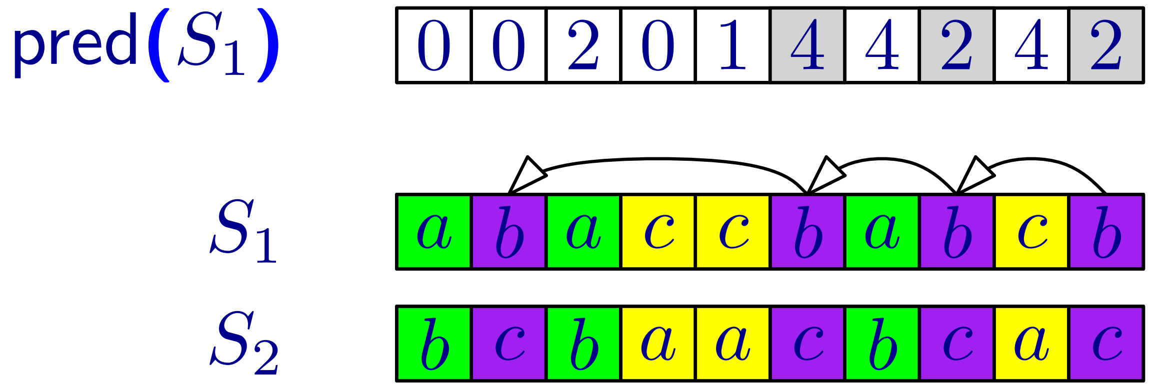

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Predecessor Strings

pred(S_1)

0	0	2	0	1	4	4	2	4	2

S_1

a	b	a	c	c	b	a	b	c	b

S_2

b	c	b	a	a	c	b	c	a	c

pred(S_2)

0	0	2	0	1	4	4	2	4	2

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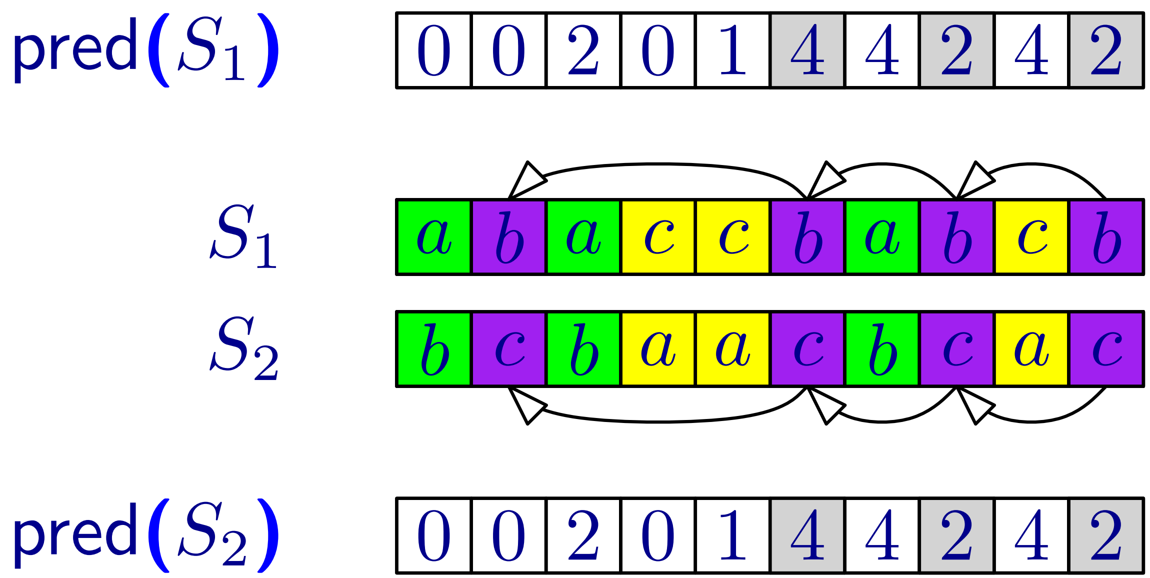

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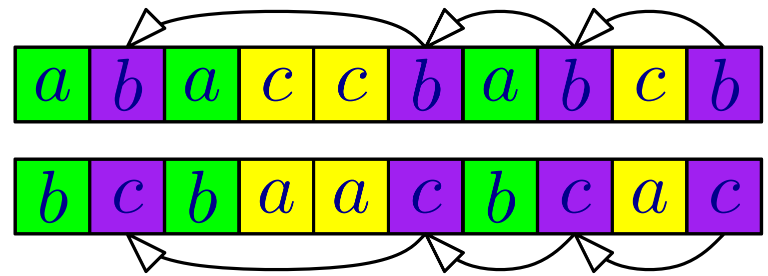
S_2



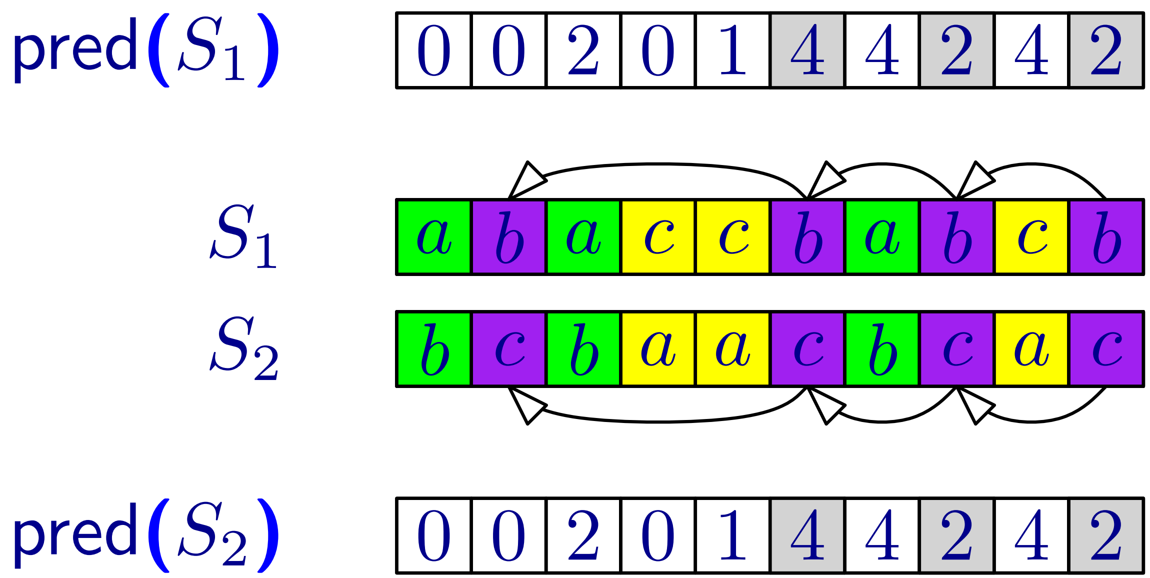
pred(S_2)

0	0	2	0	1	4	4	2	4	2

S_1 p-matches S_2 with mapping $a \to b$, $b \to c$, $c \to a$ (from S_1 to S_2)

pred(S_1)=pred(S_2)

Predecessor Strings



S_1 p-matches S_2 iff pred(S_1)=pred(S_2)

result due to Baker

Predecessor Strings

pred(T)	0	0	2	0	1	4	4	2	4	2
T	a	b	a	c	c	b	a	b	c	b
P						c	b	c	a	c

P p-matches T iff pred(P) $=$ pred$(T[i, i + |P| - 1])$

result due to Baker

Predecessor Strings

pred(T)	0	0	2	0	1	4	4	2	4	2
T	a	b	a	c	c	b	a	b	c	b
P						c	b	c	a	c
pred(P)						0	0	2	0	2

P p-matches T iff pred(P) = pred$(T[i, i + |P| - 1])$

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Predecessor Strings

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T	a	b	a	c	c	b	a	b	c	b
P						c	b	c	a	c
pred(P)						0	0	2	0	2

P p-matches T iff pred(P) = pred$(T[i, i + |P| - 1])$

result due to Baker

however, pred$(T)[i, i + |P| - 1] \neq$ pred$(T[i, i + |P| - 1])$

Predecessor Strings

pred(T)

0	0	2	0	1	4	4	2	4	2

T

a	b	a	c	c	b	a	b	c	b

P

c	b	c	a	c

pred(P)

0	0	2	0	2

P p-matches T iff pred(P) = pred$(T[i, i+|P|-1])$

result due to Baker

however, pred$(T)[i, i+|P|-1] \neq$ pred$(T[i, i+|P|-1])$

some values may have to be *zeroed*

Rabin-Karp fingerprints of strings

S

a	b	a	c	c	b	a	b	c	b

$$\phi(S) = \sum_{k=0}^{|S|-1} S[k]r^k \bmod p$$

Here $p = \Theta(|T|^4)$ is a prime and $1 \leq r < p$ is a random integer

with high probability, $S_1 = S_2$ iff $\phi(S_1) = \phi(S_2)$

Rabin-Karp fingerprints of strings

S | a | b | a | c | c | b | a | b | c | b

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Observe that $\phi(S)$ fits in an $O(\log |T|)$ bit word

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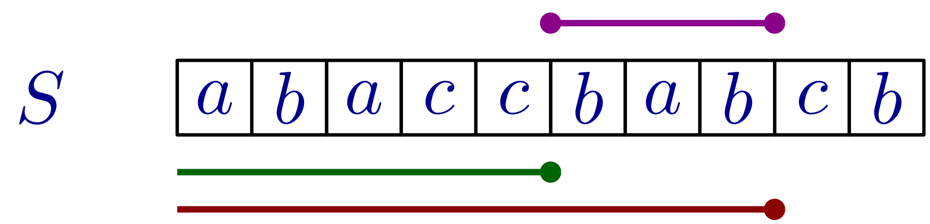

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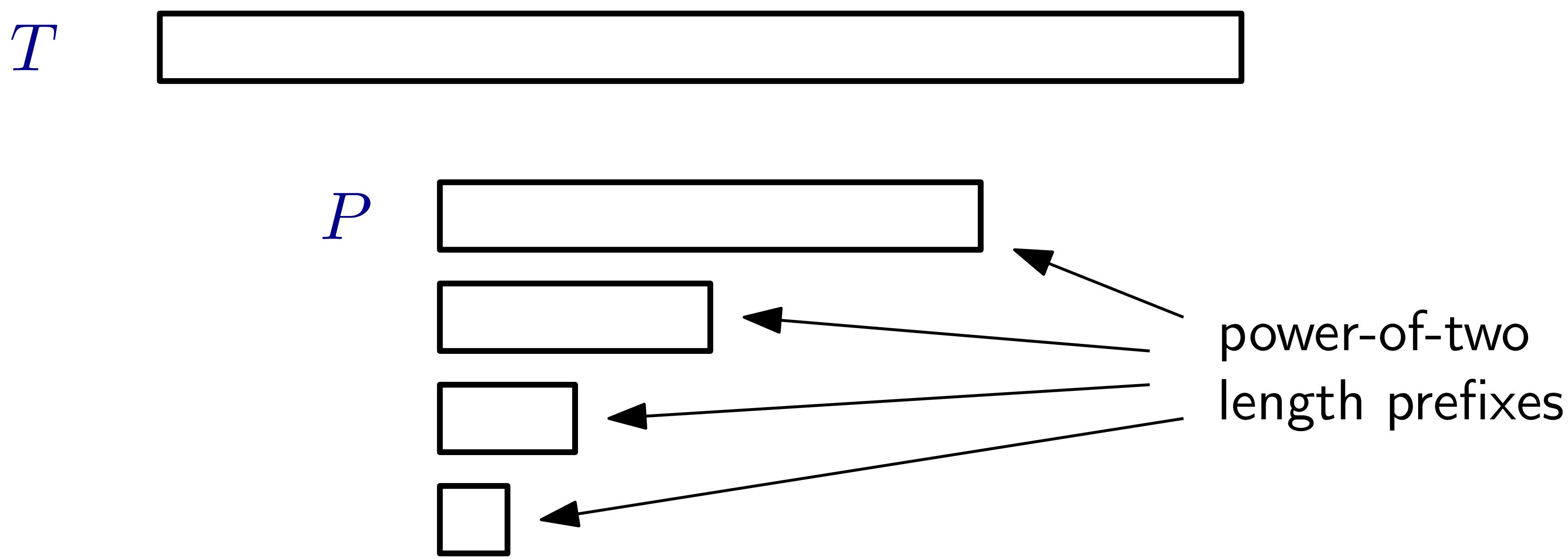
Given $\phi(S[0,\ell])$ and $\phi(S[0,r])$ we can compute $\phi(S[\ell+1,r])$ in $O(1)$ time

Exact matching using fingerprints (Porat and Porat)



Find matches with each power-of-two length prefix

Exact matching using fingerprints (Porat and Porat)

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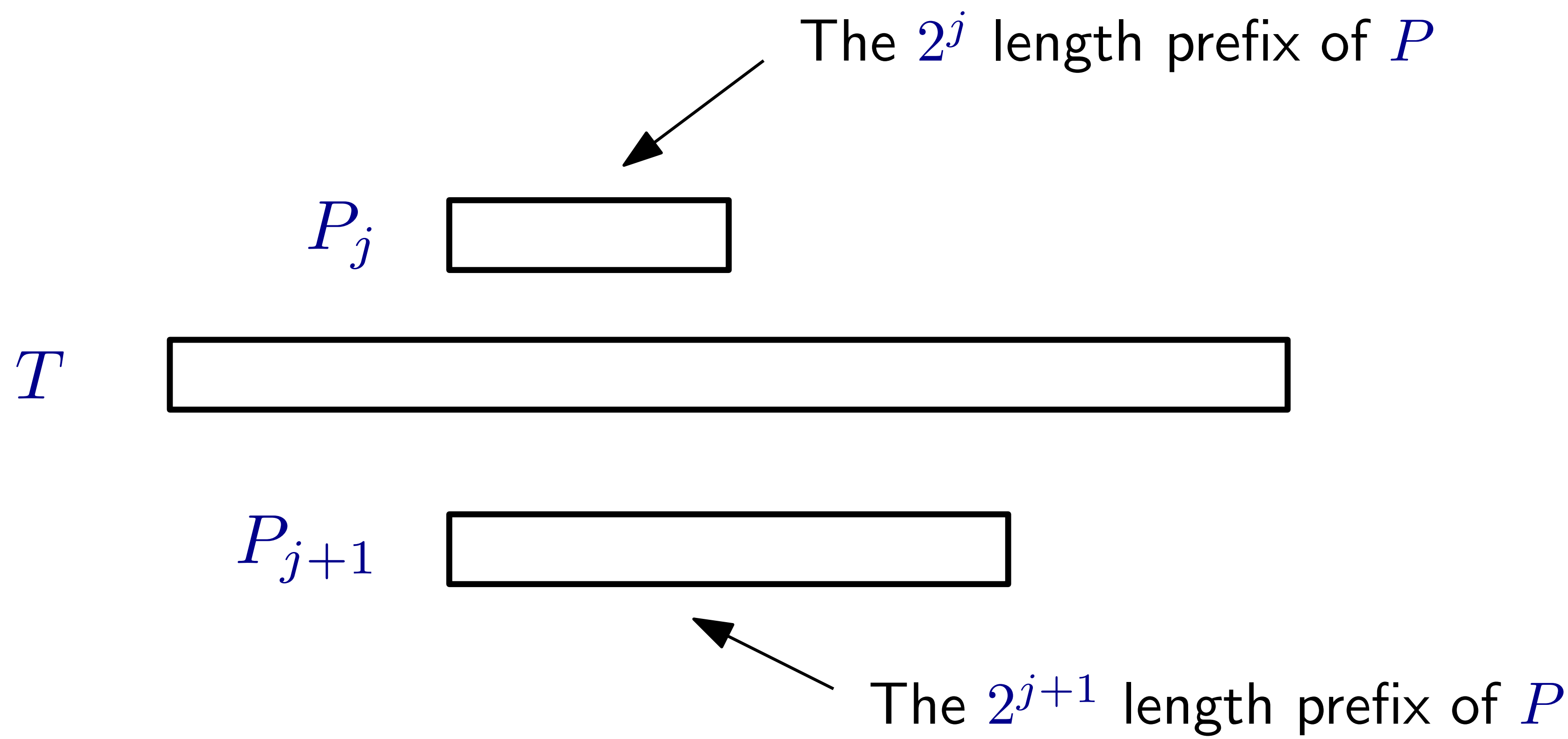

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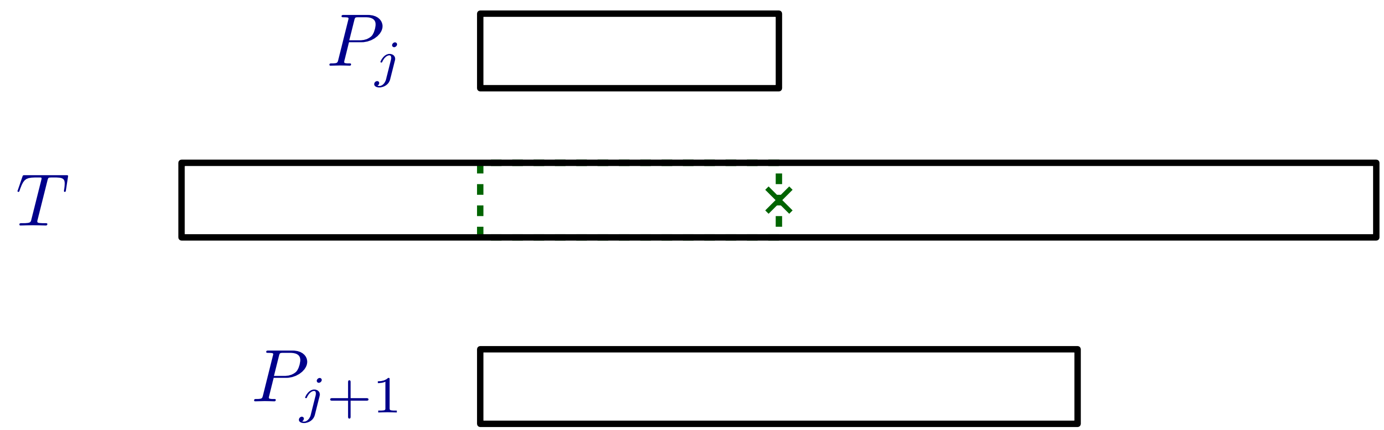

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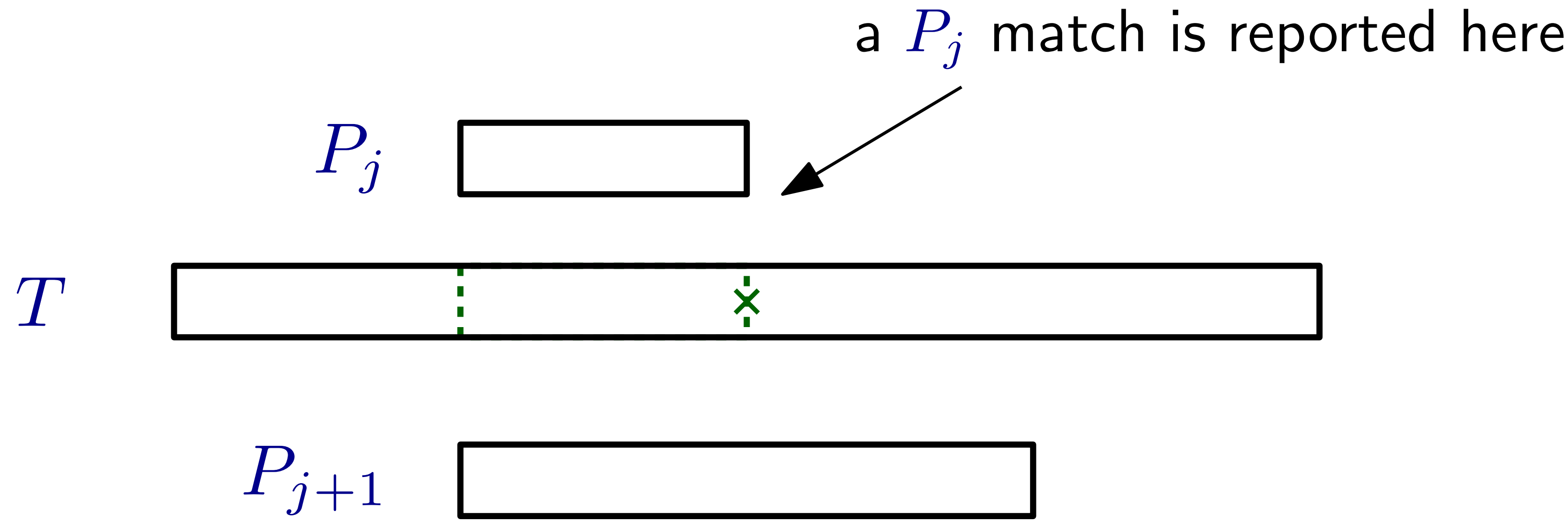

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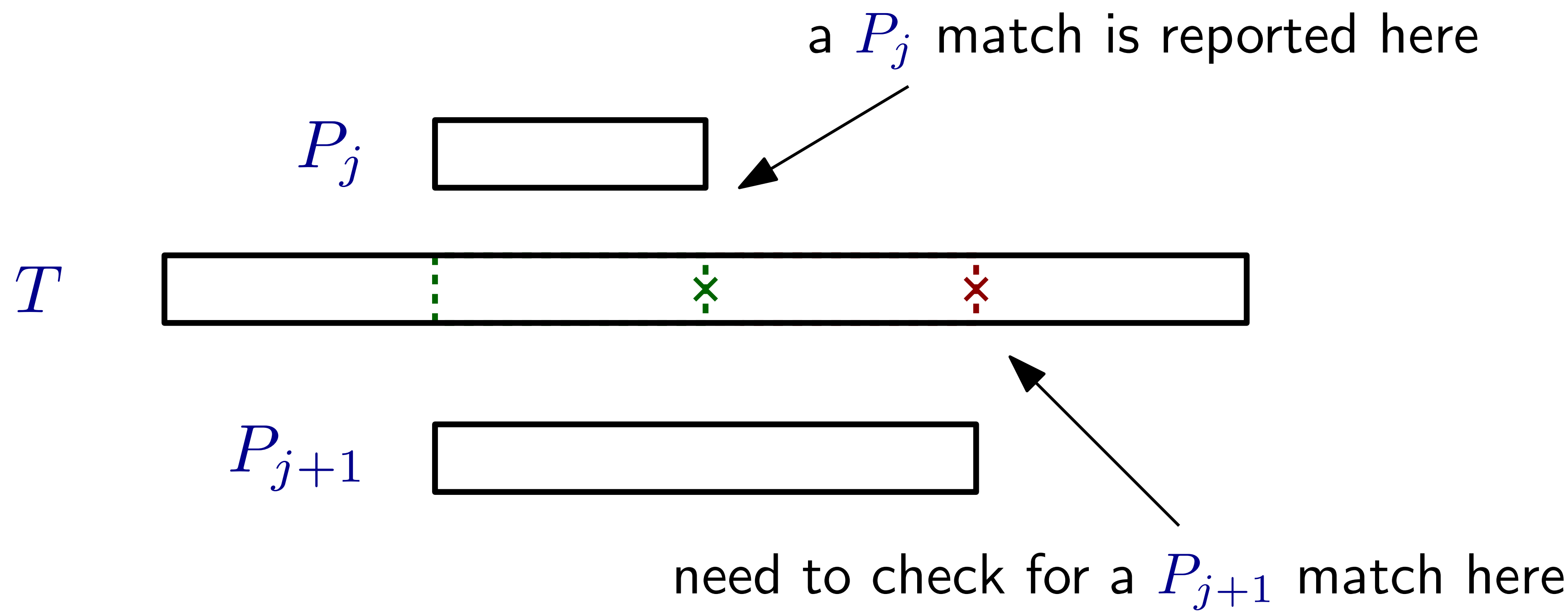

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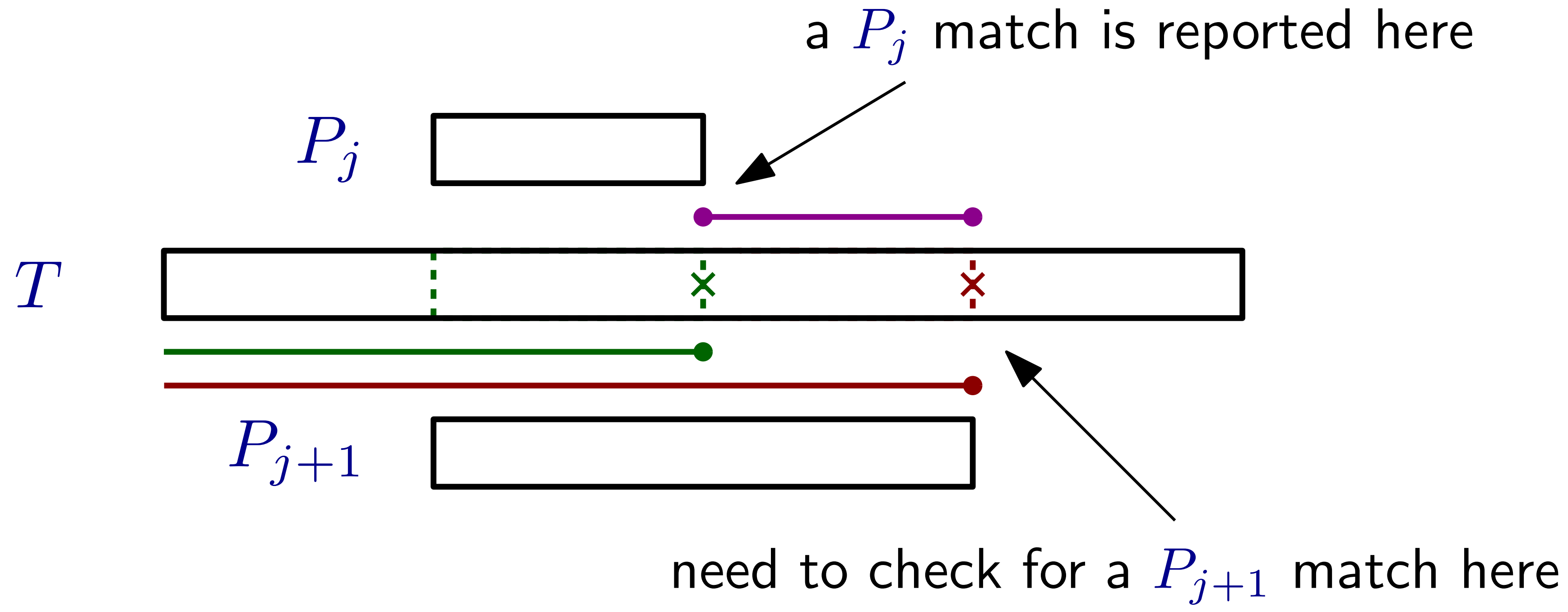

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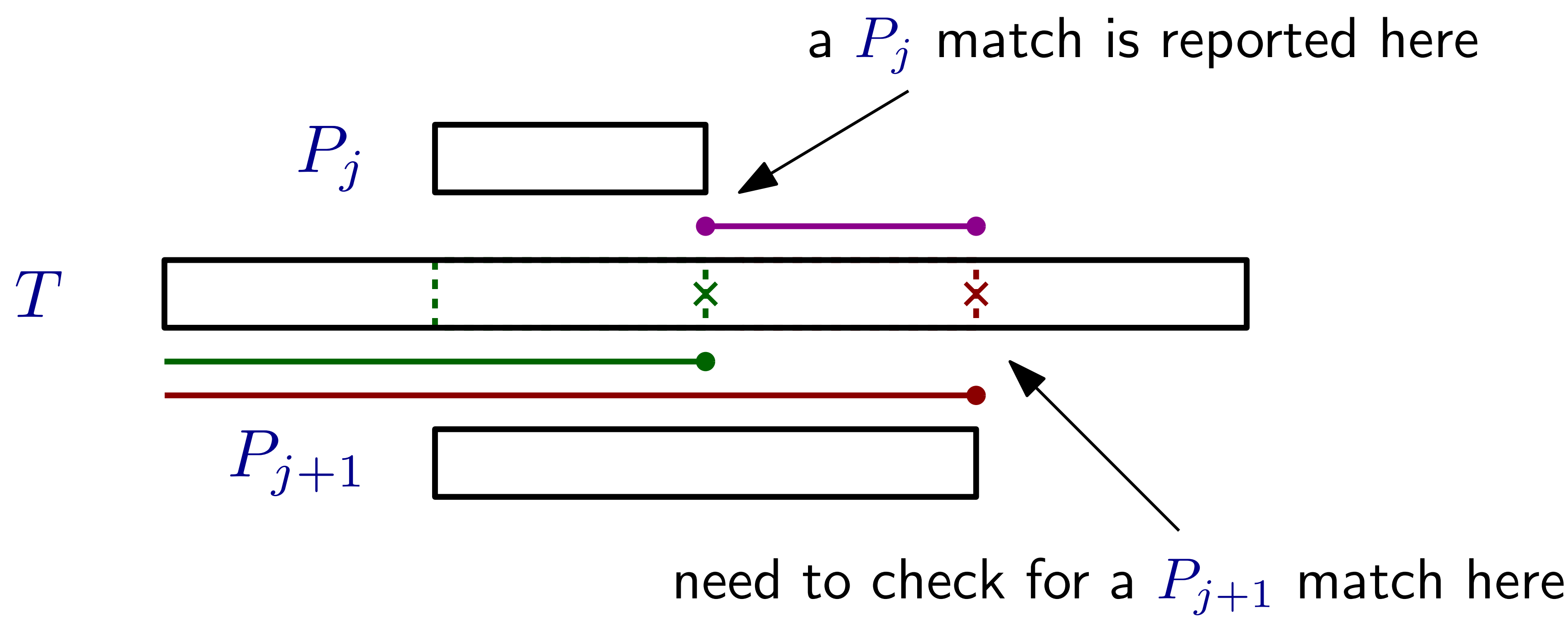

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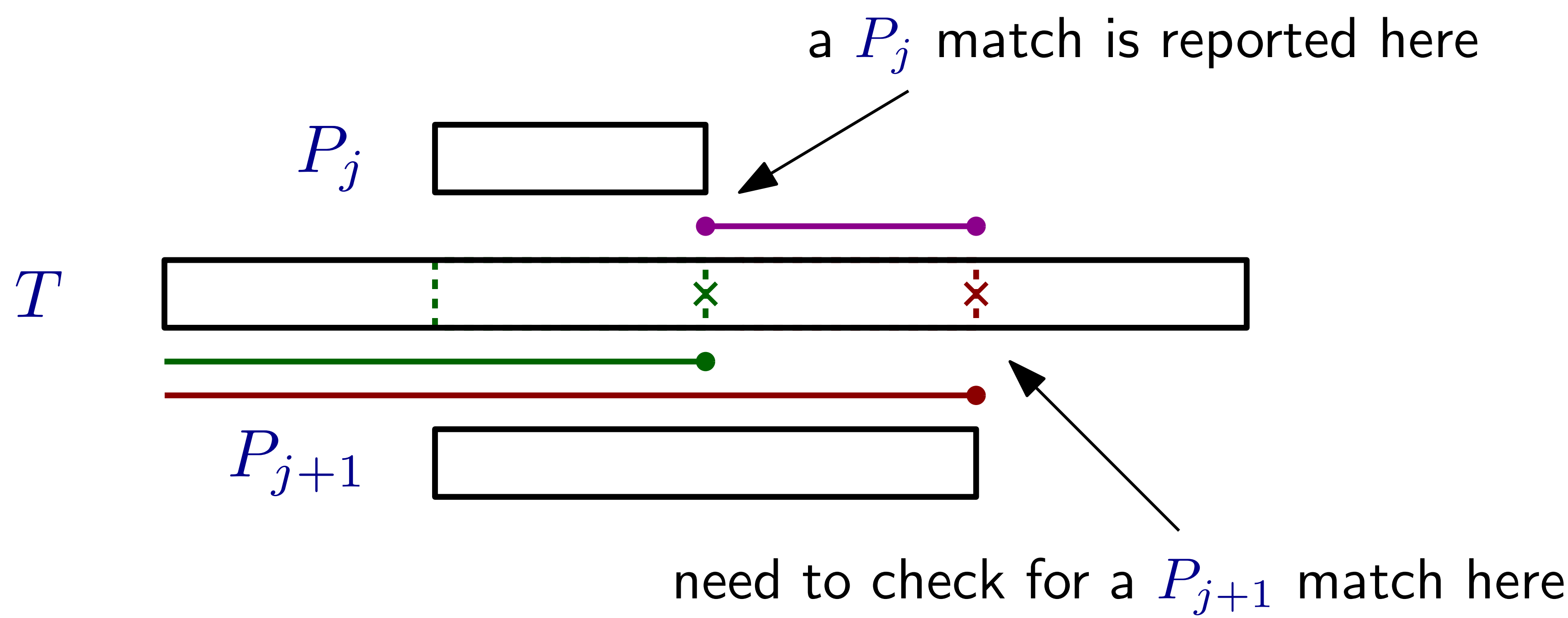
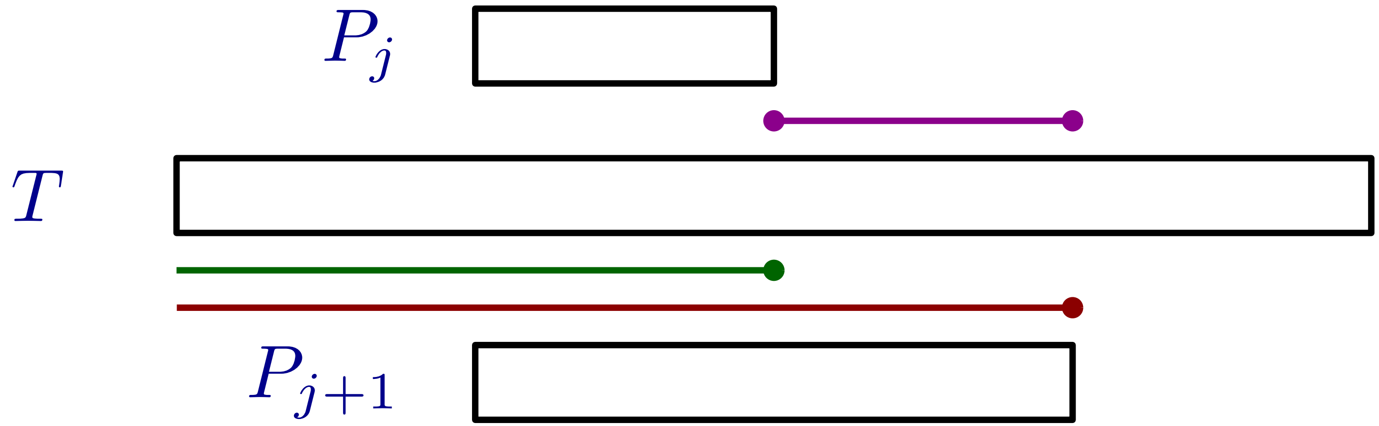


Given $\phi(T[0,\ell])$ and $\phi(T[0,r])$ we can compute $\phi(T[\ell+1,r])$ in $O(1)$ time

To decide whether P_{j+1} matches,
compare $\phi(T[\ell+1,r])$ to $\phi(P[2^j + \ldots 2^j - 1])$

Exact matching using fingerprints (Porat and Porat)



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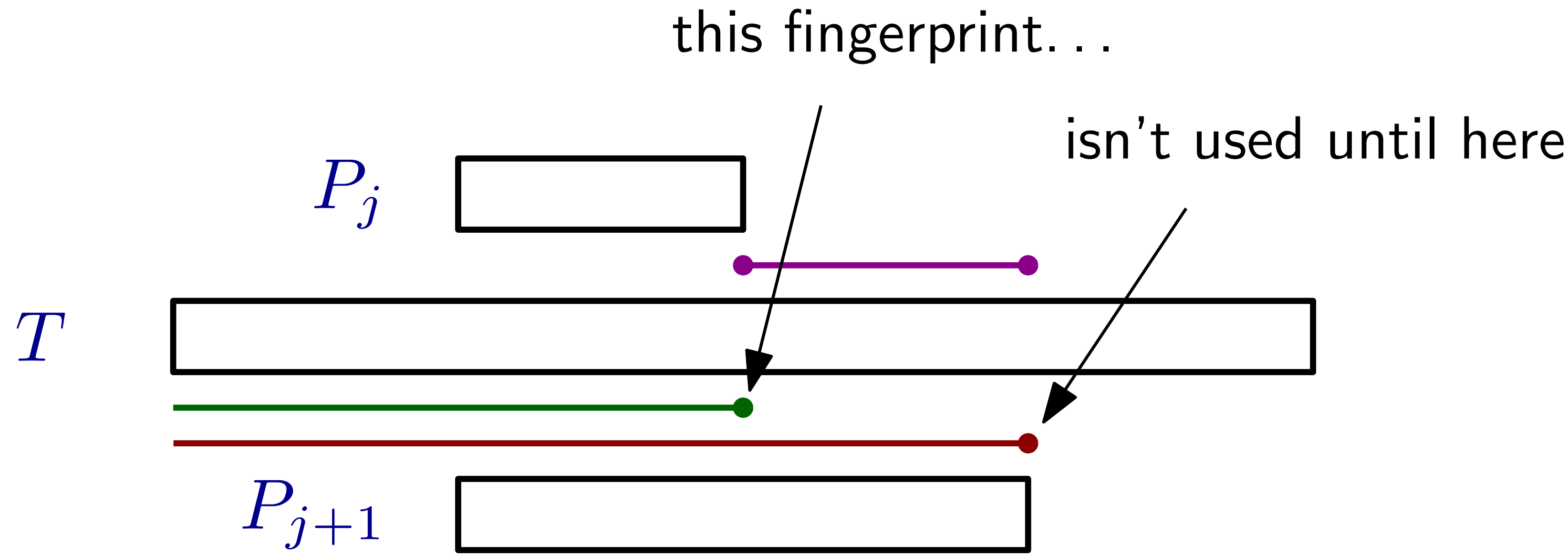

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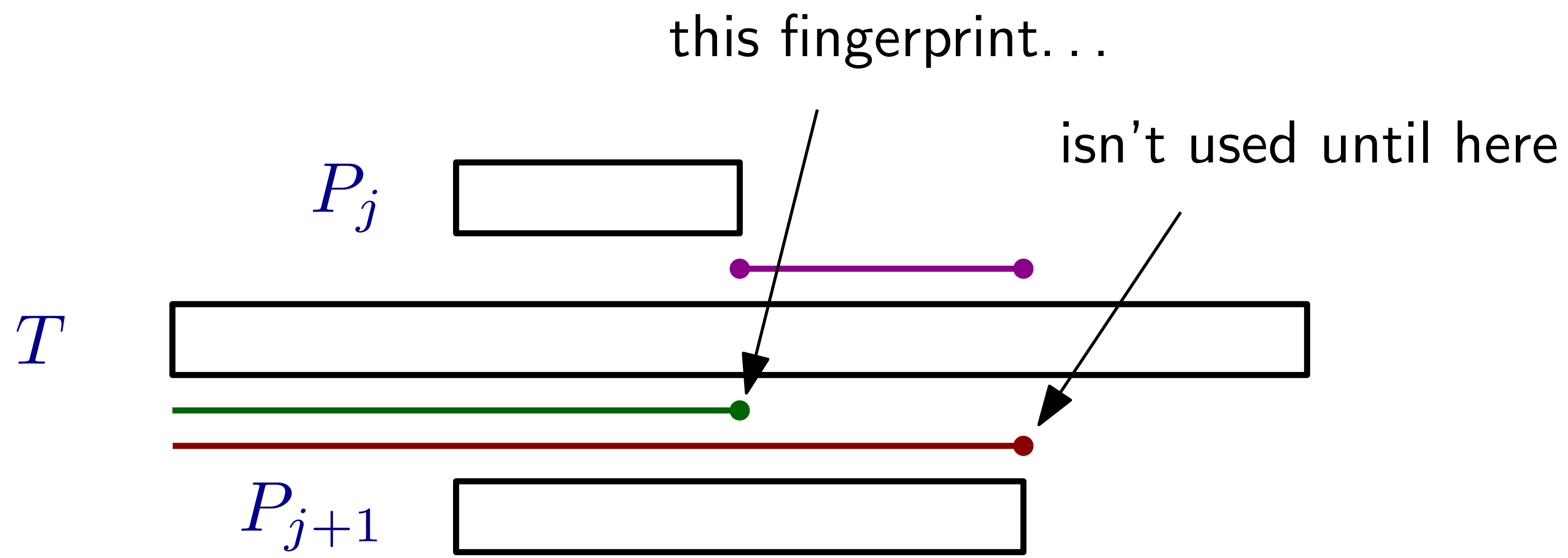

Each ϕ fits in a word but we need to store each one for a long time...

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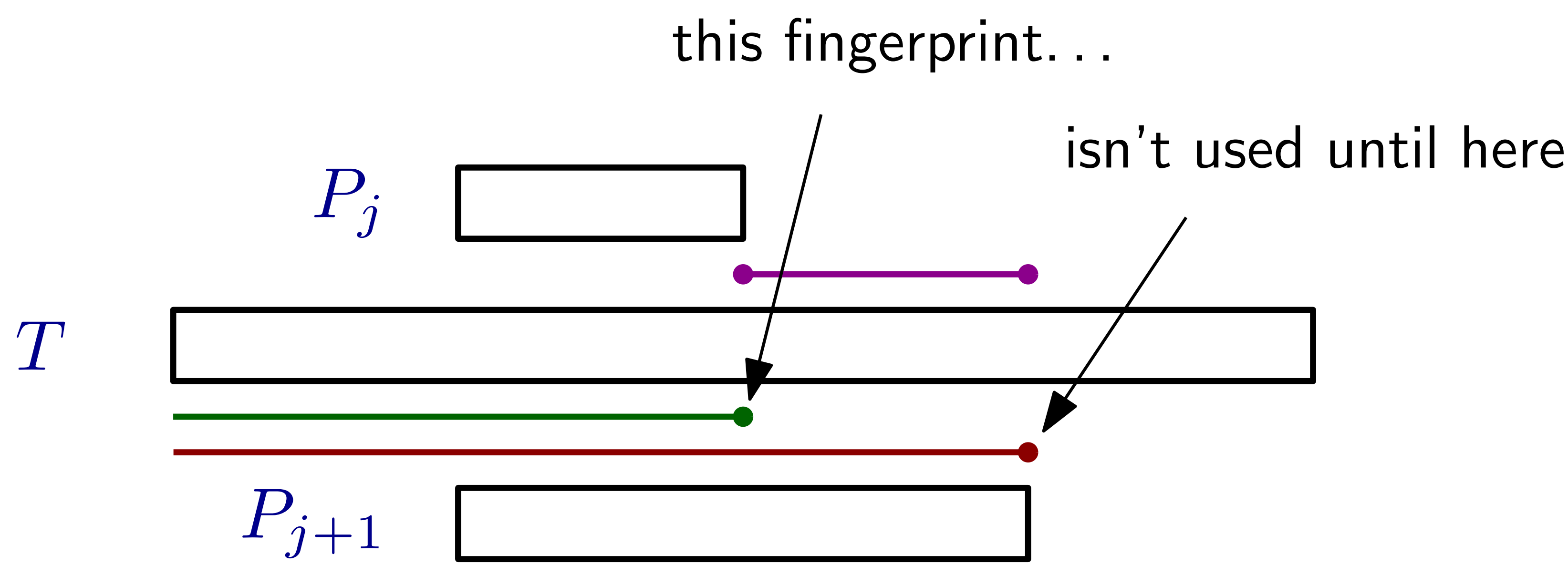

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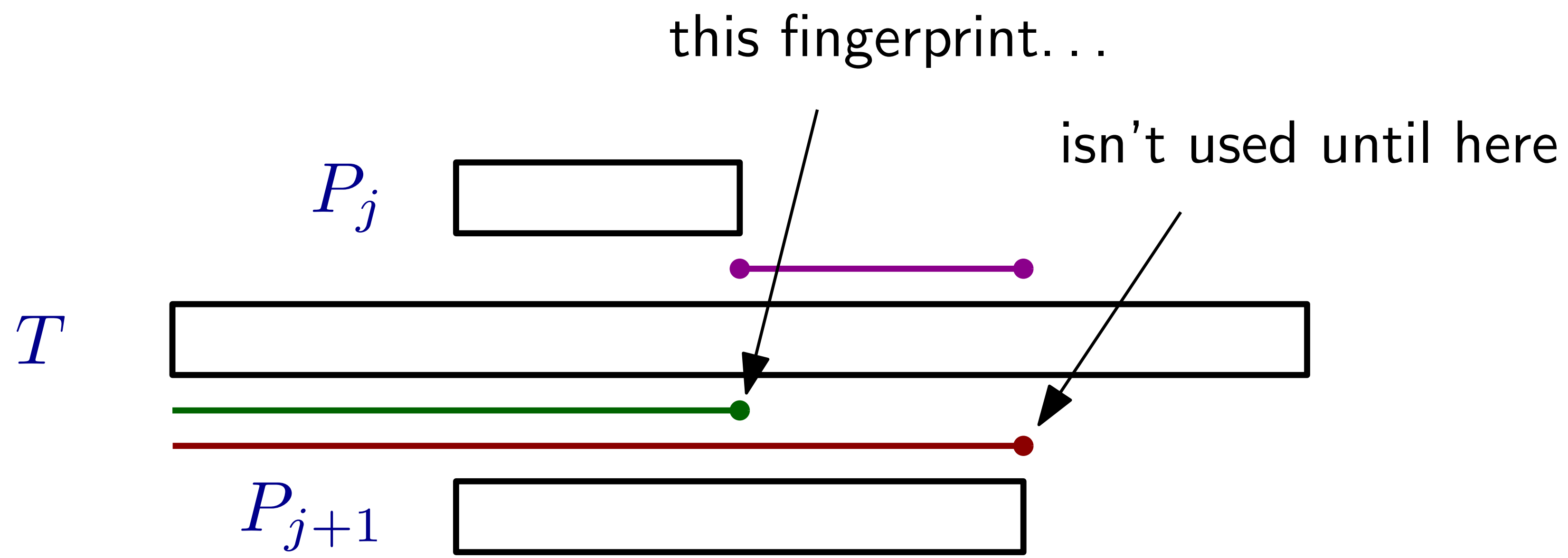
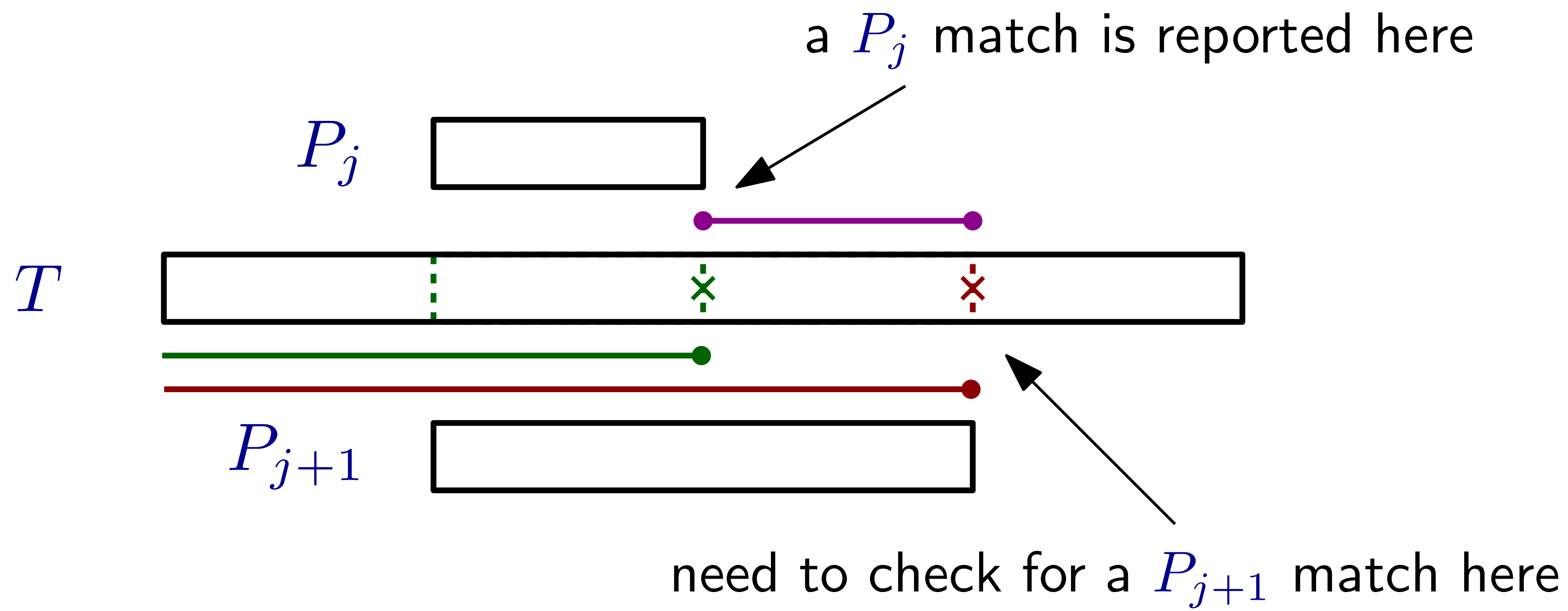


Each ϕ fits in a word but we need to store each one for a long time...
we may have to store many ($\Omega(|P|)$) at a time.

the fingerprints themselves have to be stored in a compressed form

Para. matching using fingerprints (this paper)



Overall approach: Find matches using fingerprints of predecessor strings

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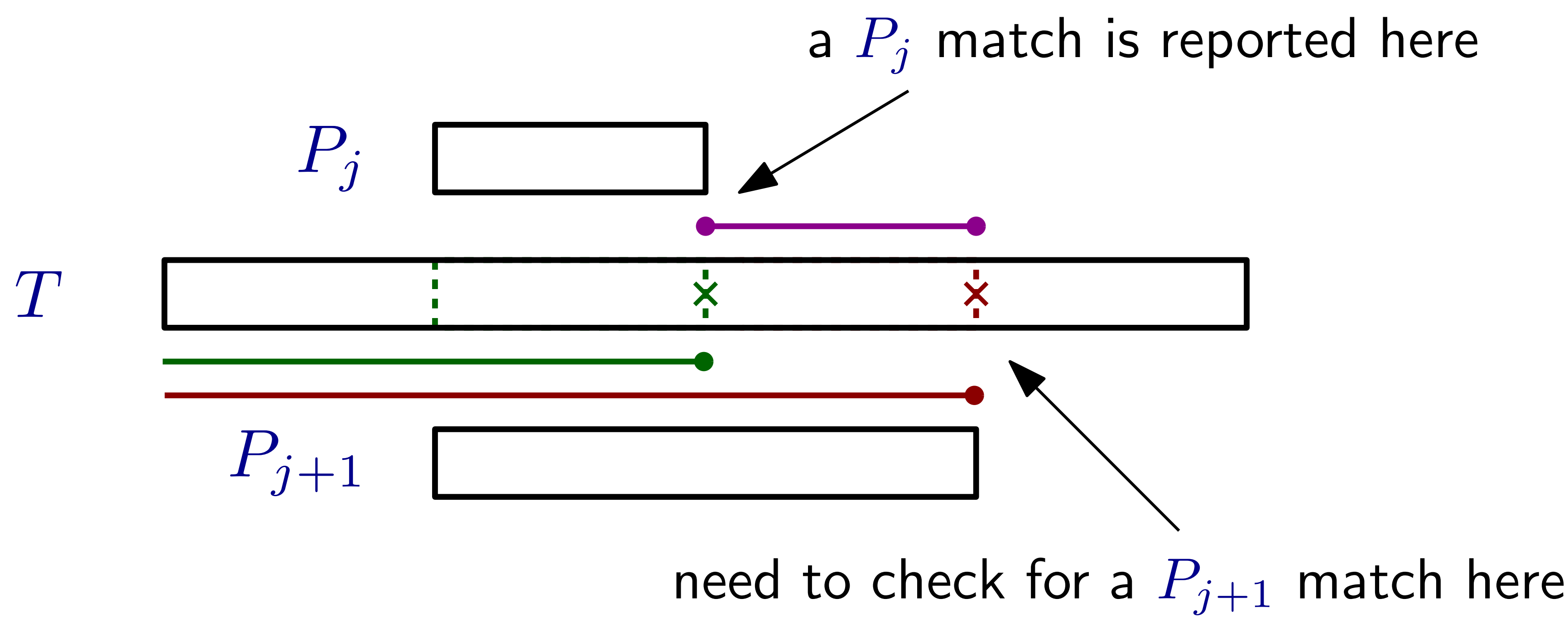


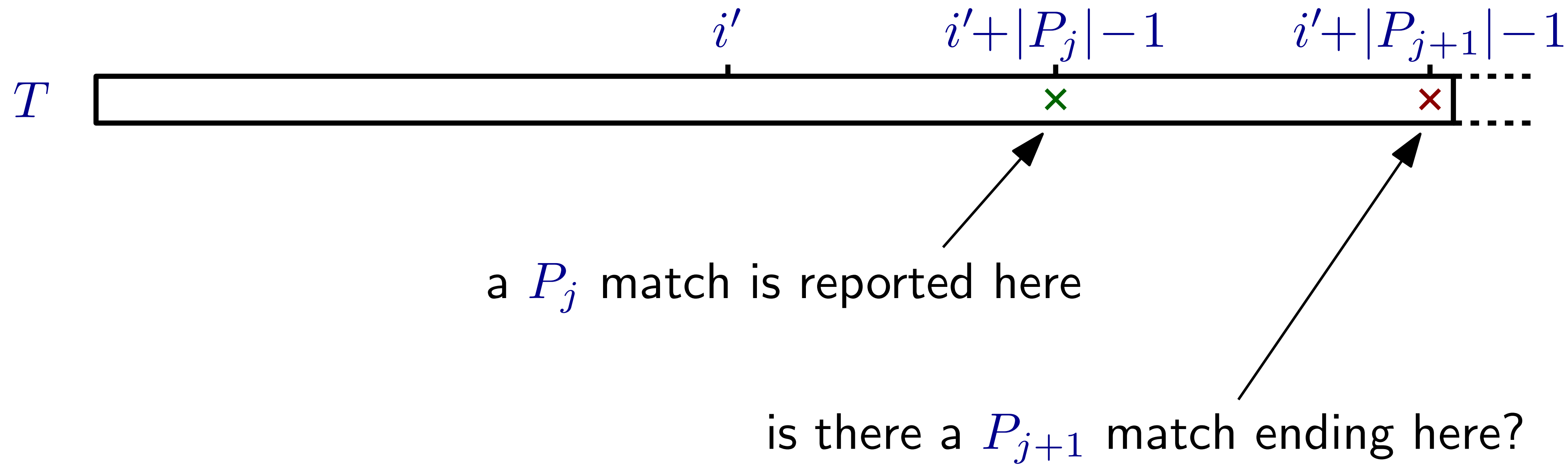
Overall approach: Find matches using fingerprints of predecessor strings

Key Problem 1: $\mathrm{pred}(T)[\ell+1, r] \neq \mathrm{pred}(T[\ell+1, r])$

Key Problem 2: How do we store all the fingerprints?

Key Problem 3: How do we deamortise the algorithm?

Key Problem 1: Correcting predecessor fingerprints



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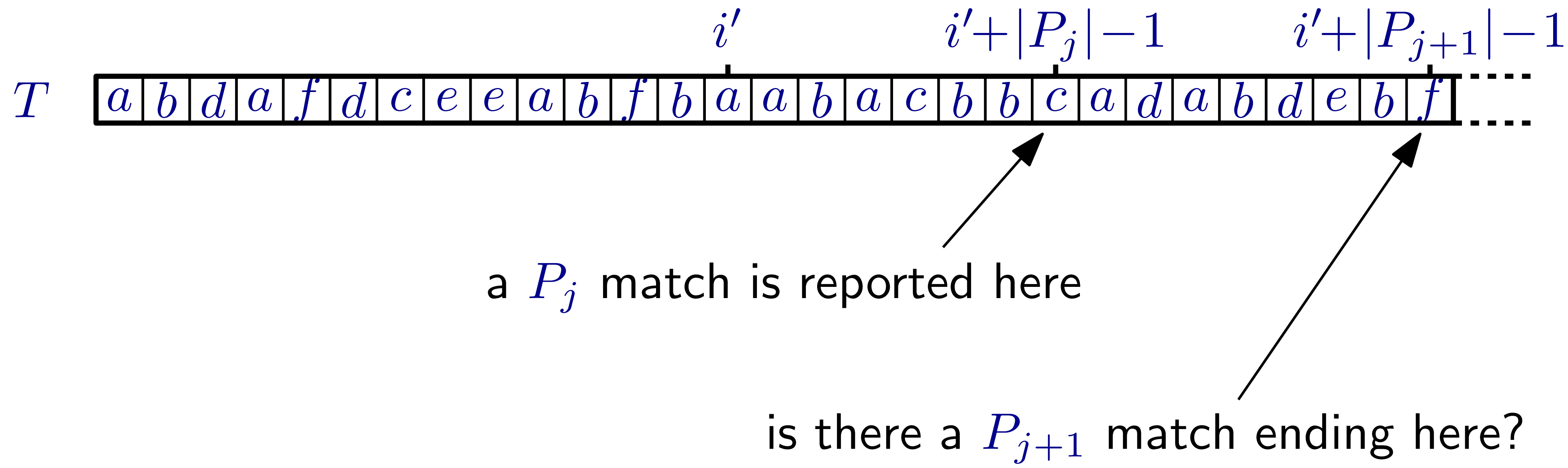

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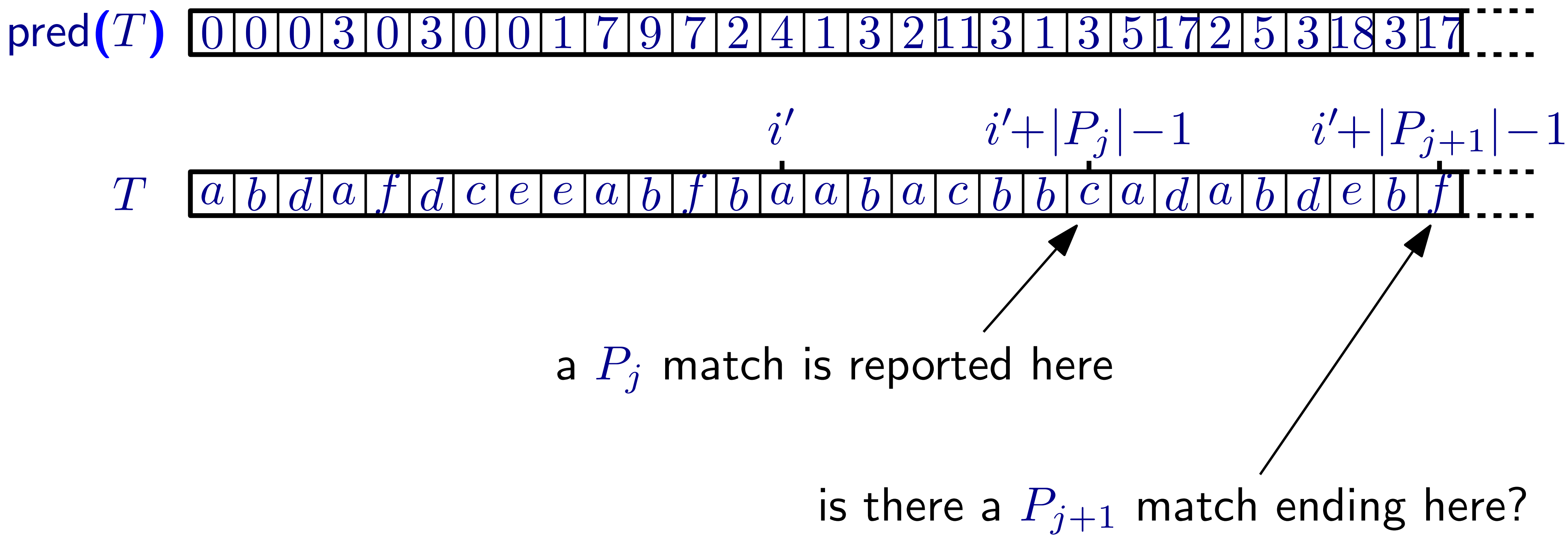

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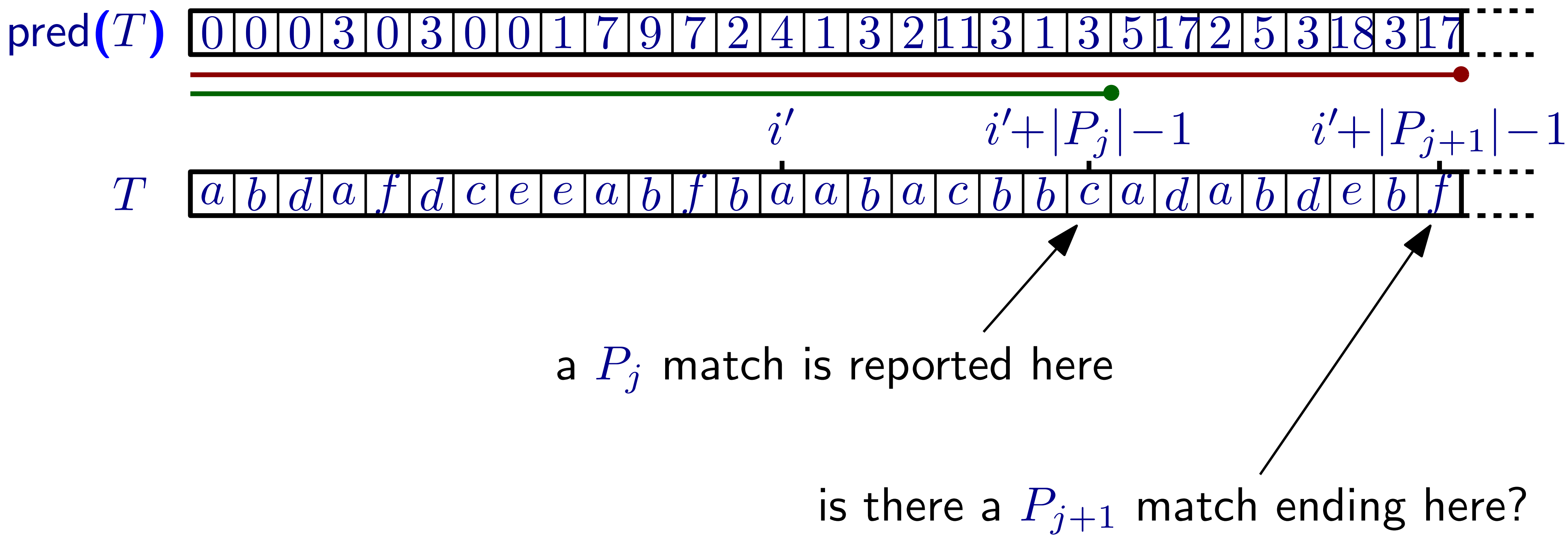

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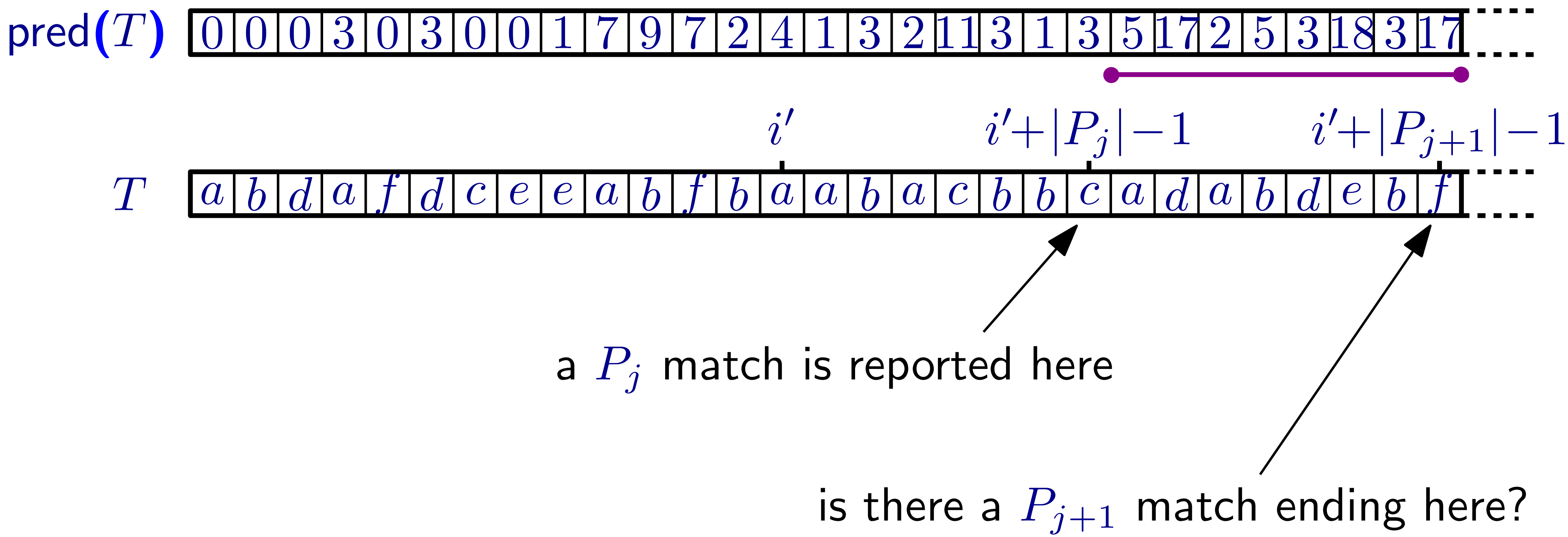

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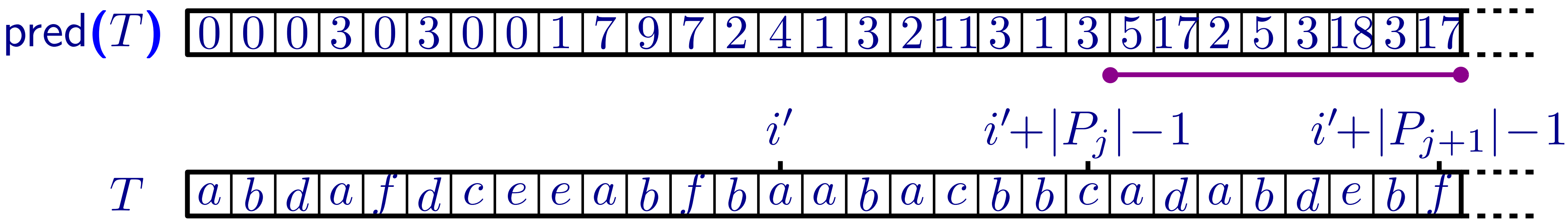

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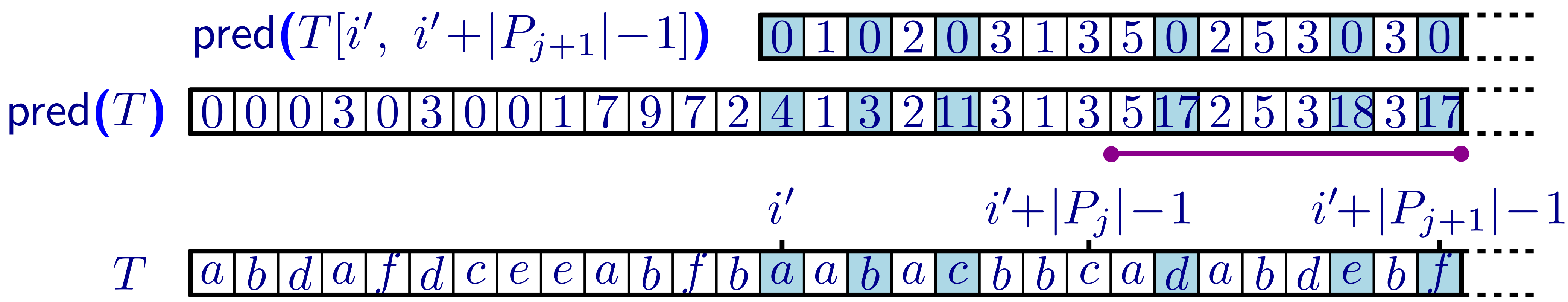

P_{j+1} p-matches iff

pred($T[i', i'+|P_{j+1}|-1]$) = pred(P_{j+1})

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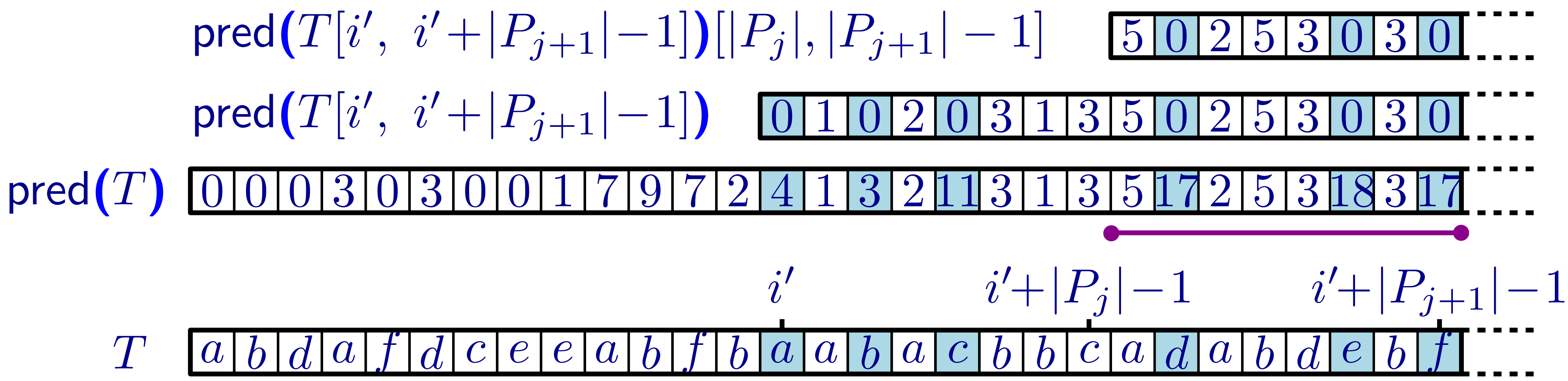

As P_j p-matches,

P_{j+1} p-matches iff

$$\text{pred}(T[i', i'+|P_{j+1}|-1])[|P_j|, |P_{j+1}|-1] = \text{pred}(P_{j+1})[|P_j|, |P_{j+1}|-1]$$

Compute this online

Precompute this

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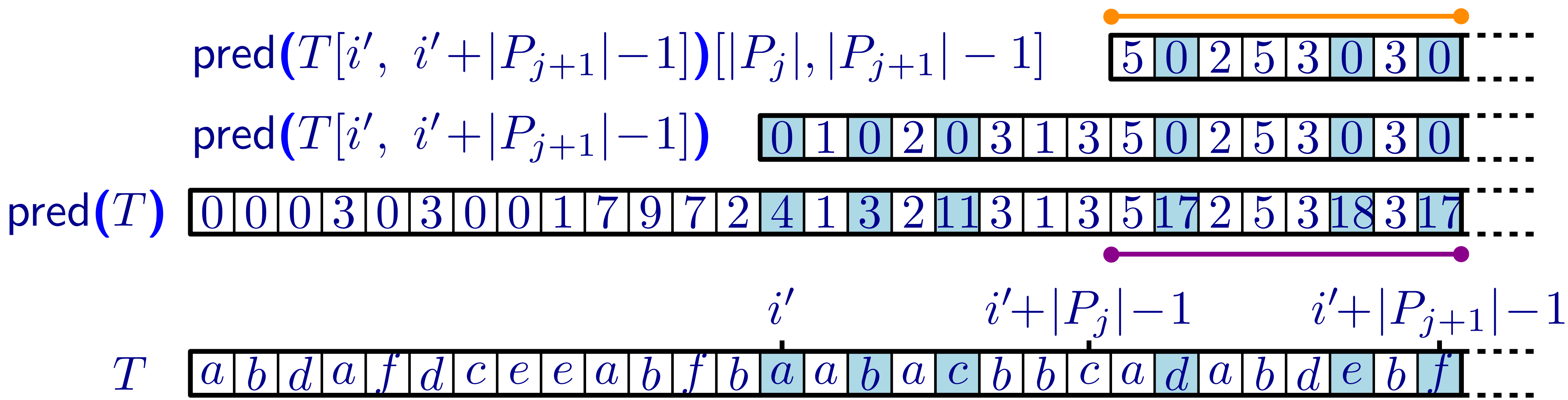

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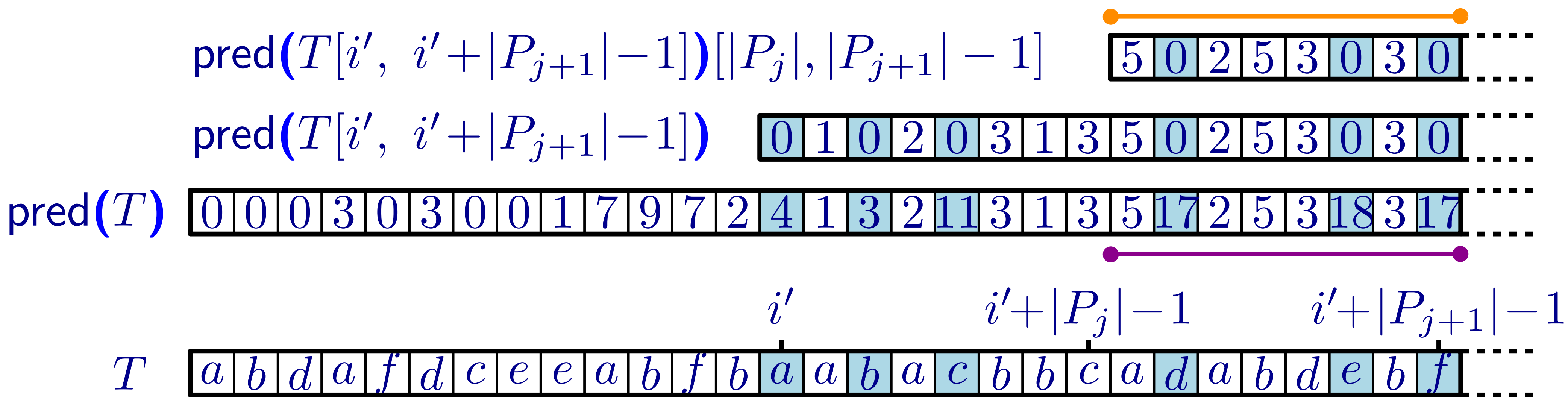

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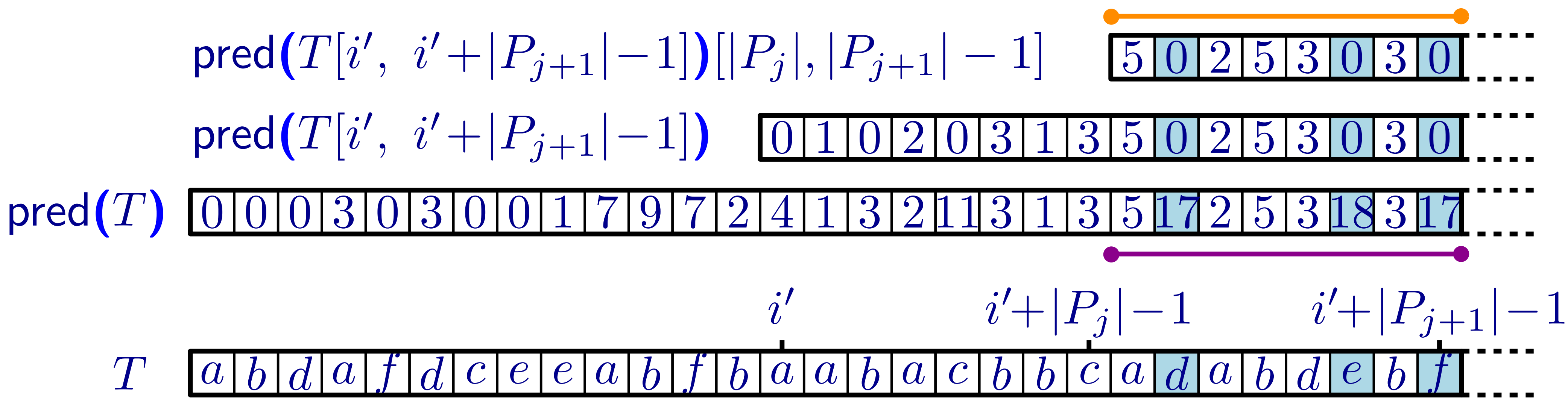

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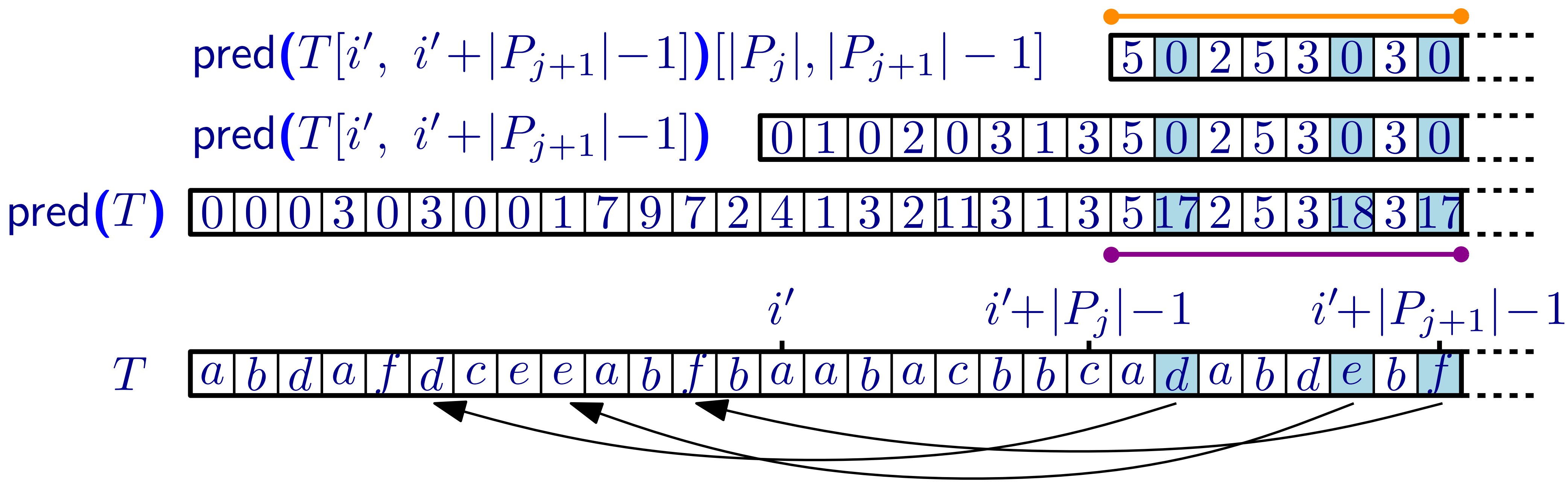

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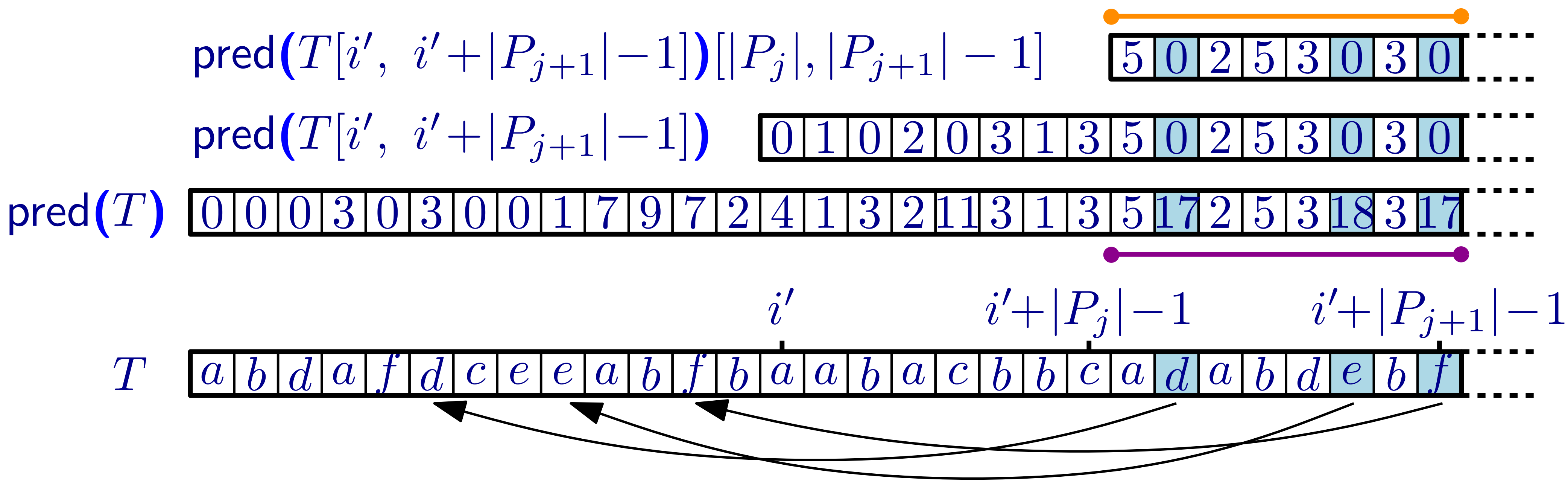

All these characters have large predecessor values. . . at least $|P_j|$

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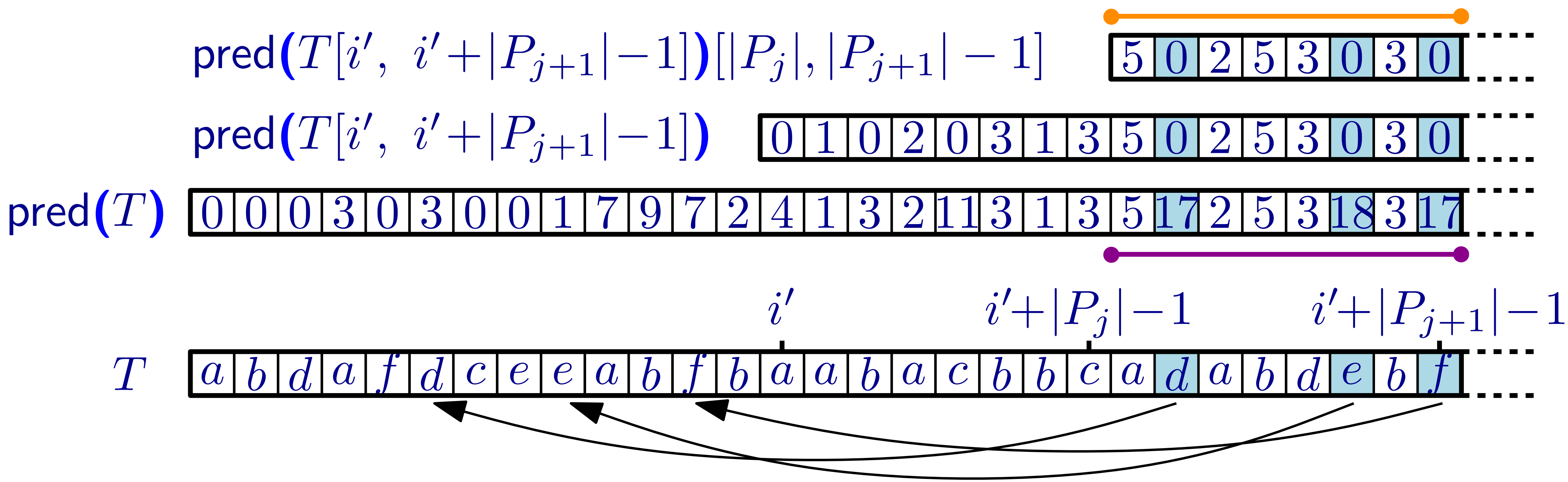

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There are $O(|\Sigma|)$ such characters in an $O(|P_{j+1}|)$ length text window. . .
so we can store them all

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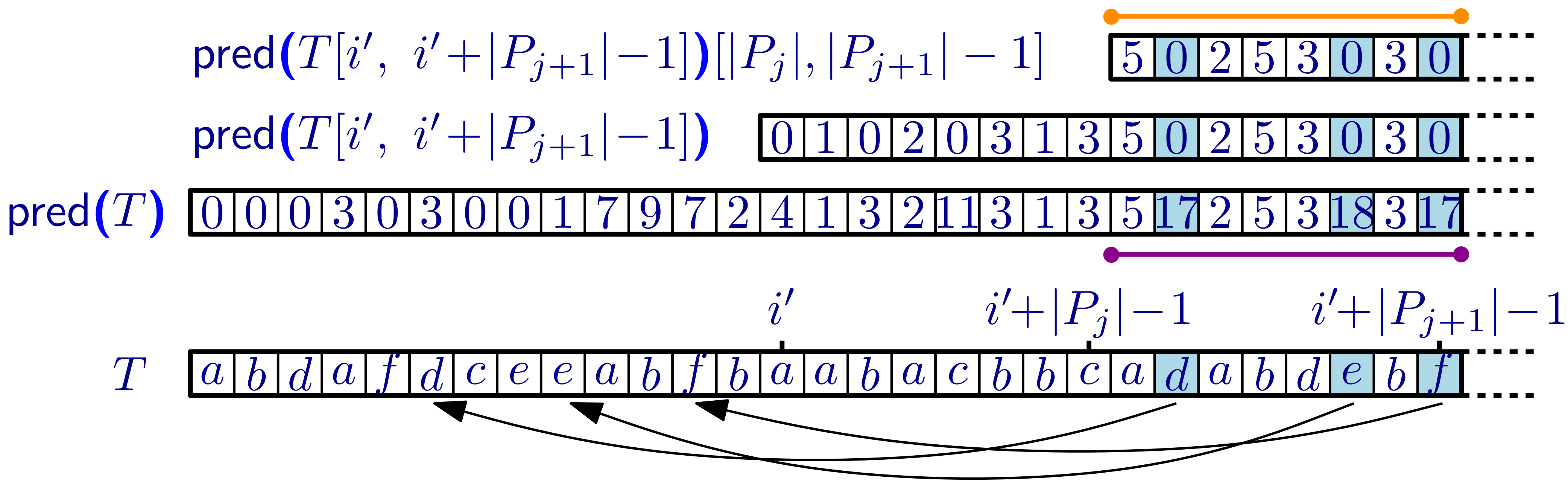

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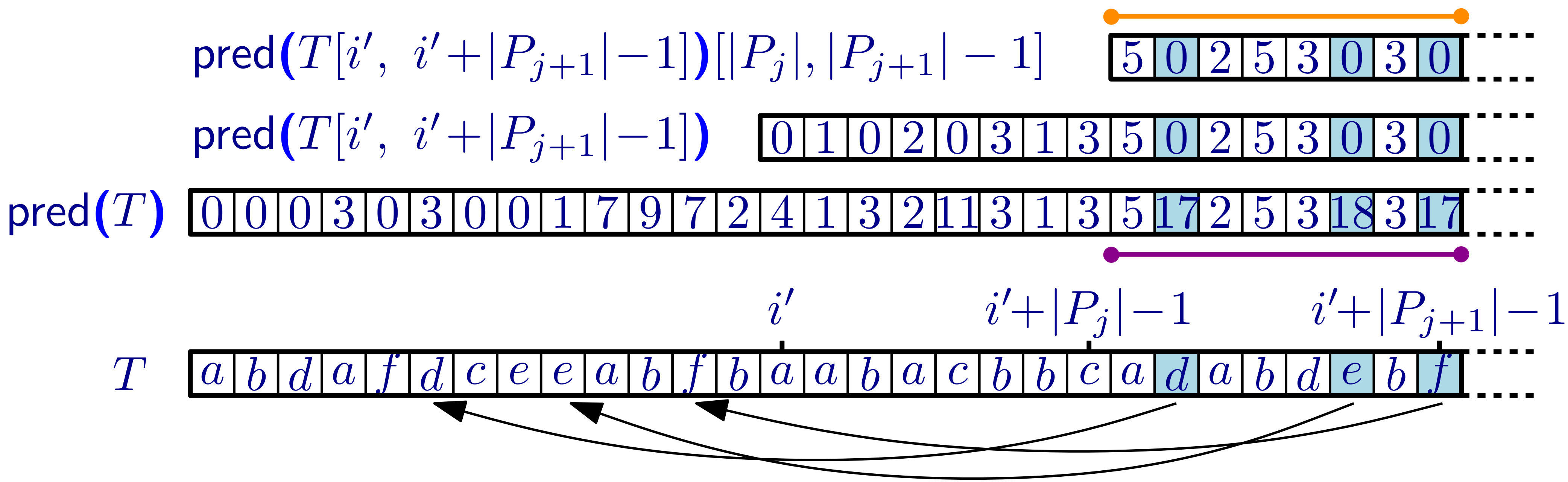

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Modifying the fingerprint in $O(|\Sigma|)$ time is simple arithmetic...
don't panic about the time complexity - we'll fix that later

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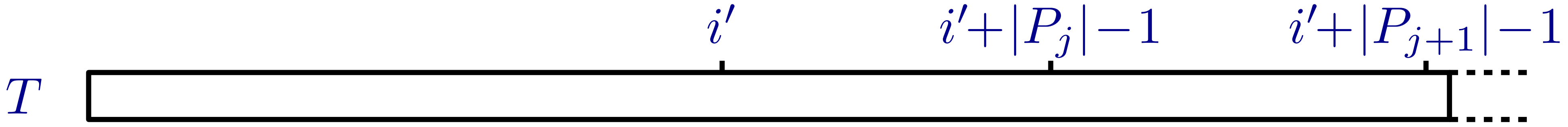

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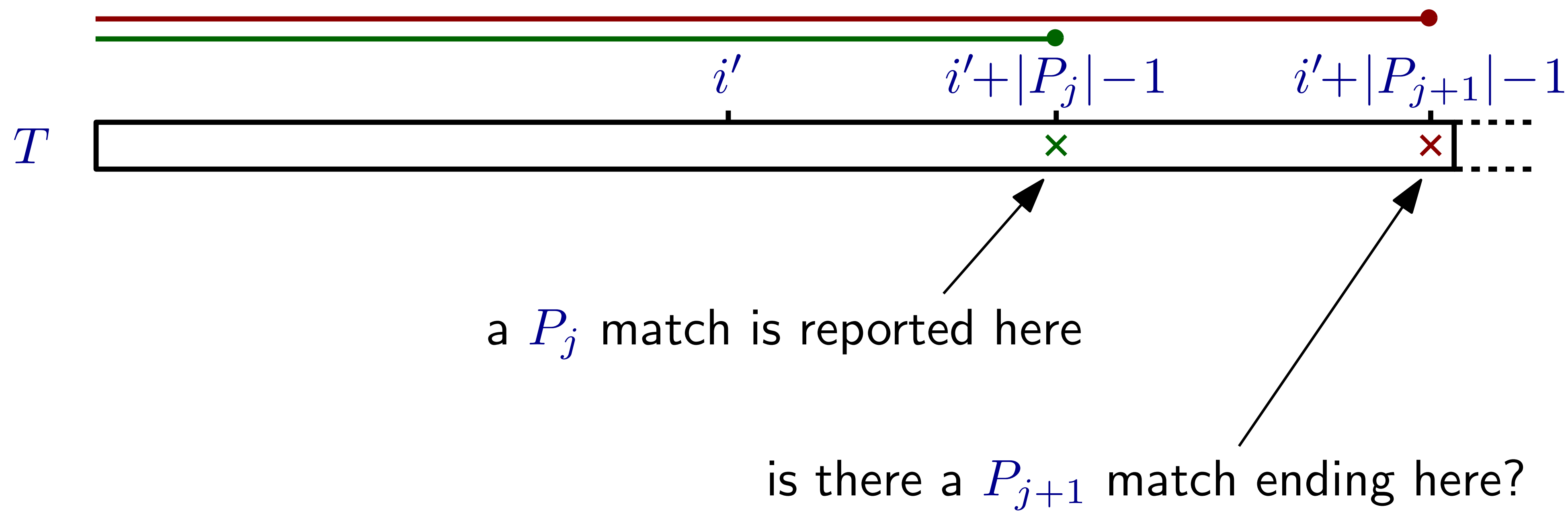

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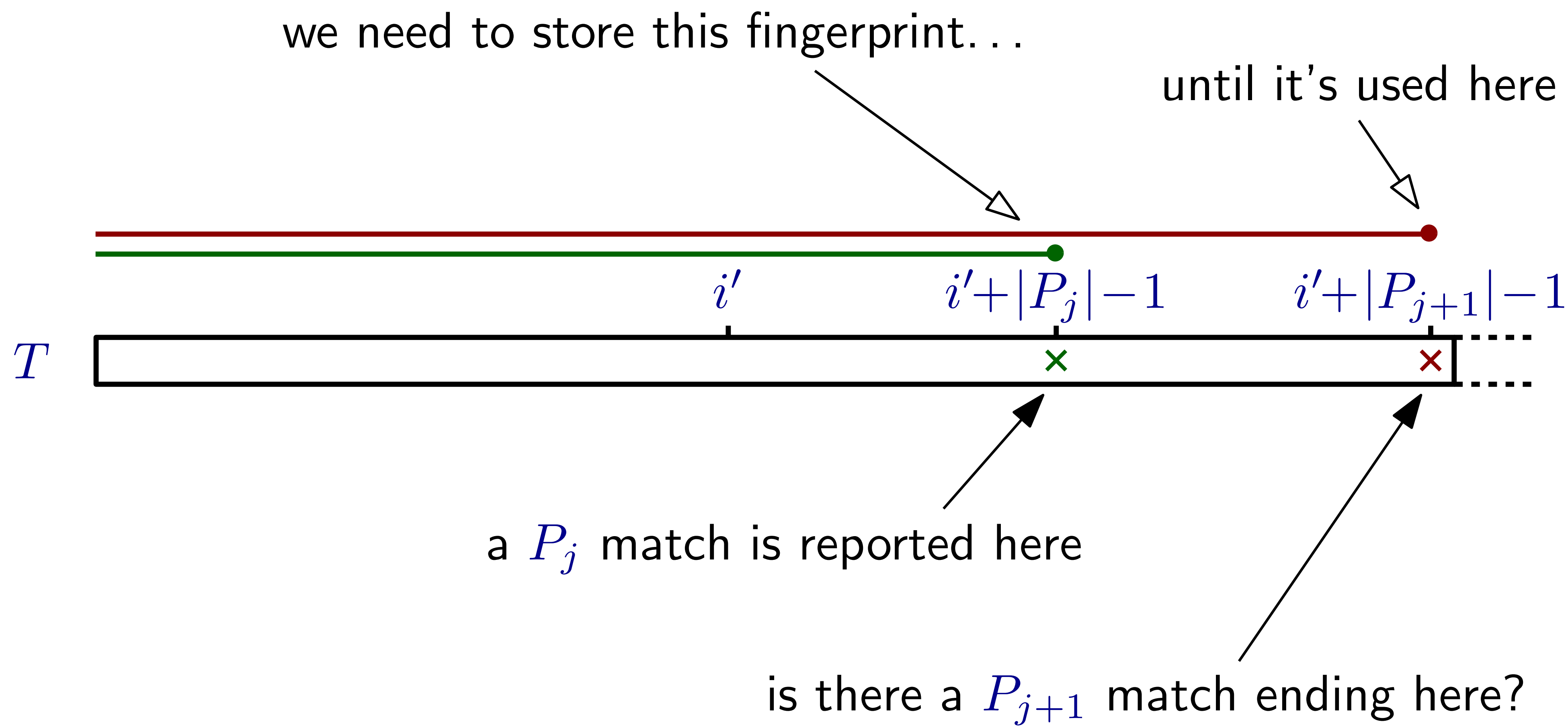
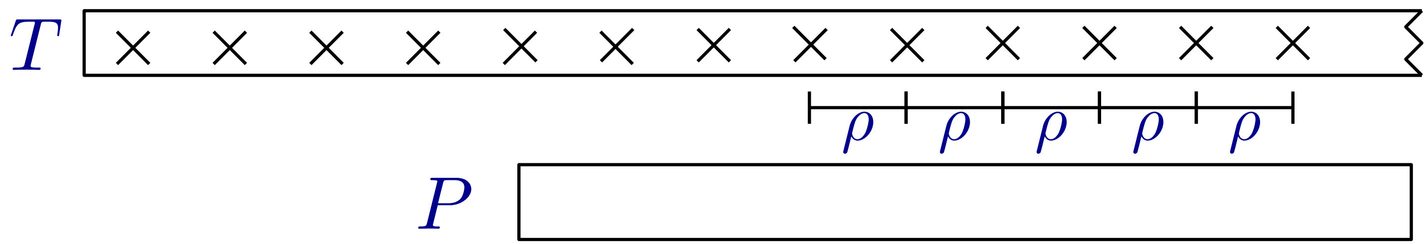


Key Problem 2: Storing the fingerprints

The structure of exact matches

either every ρ or far apart



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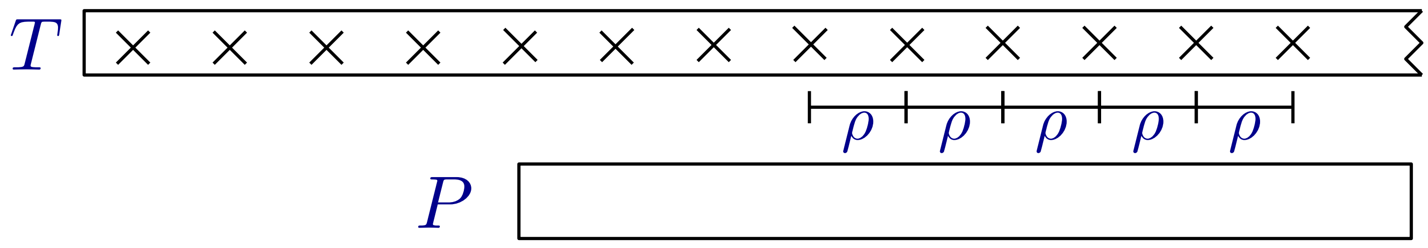

this allowed the partial **exact** *matches with each* P_j *to be encoded as an arithmetic progression in constant space*

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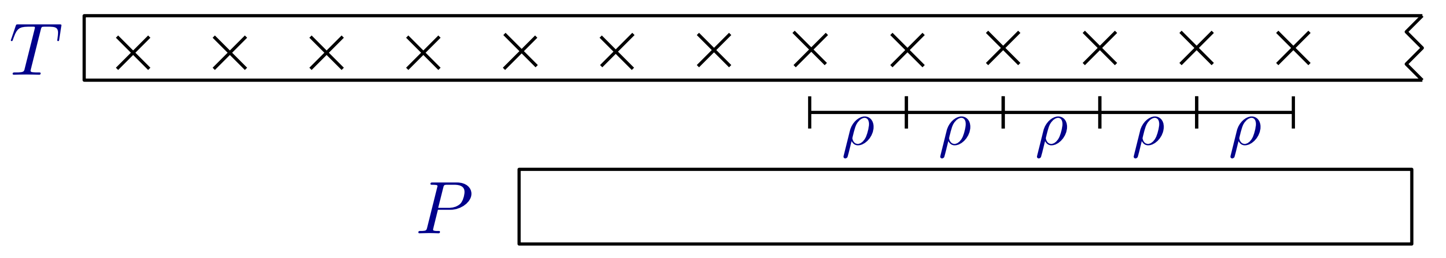
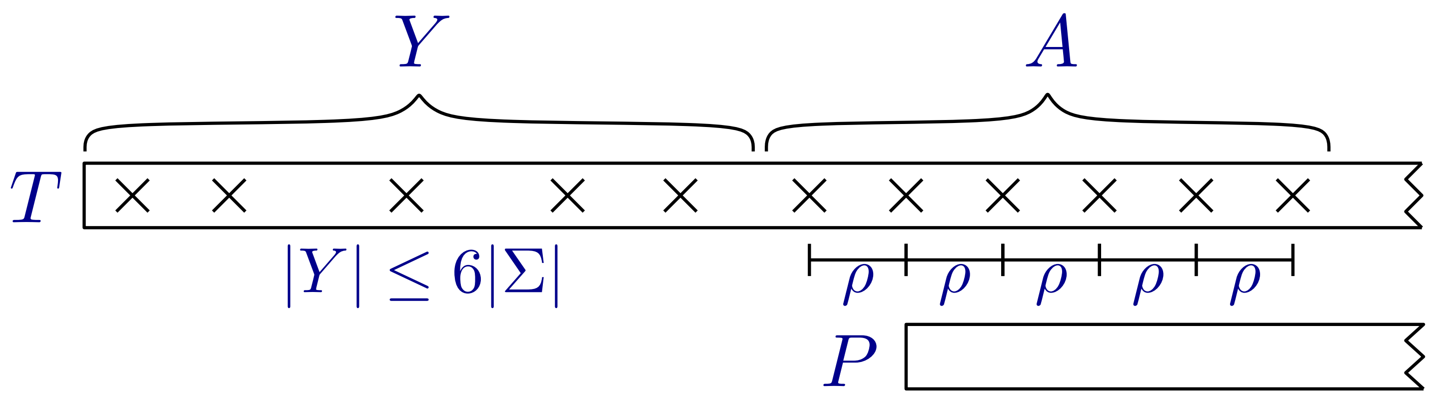


this allowed the partial **exact** *matches with each P_j to be encoded as an arithmetic progression in constant space*

the fingerprints can also be encoded in an analagous manner

Key Problem 2: Storing the fingerprints

The structure of parameterized matches



this allows the partial parameterized matches with each P_j to be encoded in $O(|\Sigma|)$ space

the fingerprints can also be encoded in an analagous manner

Key Problem 3: Deamortising the algorithm

As described, the algorithm takes $O(|\Sigma|)$ time per prefix match found.

Using an idea of Breslauer and Galil, this is reduced to $O(1)$ per char.

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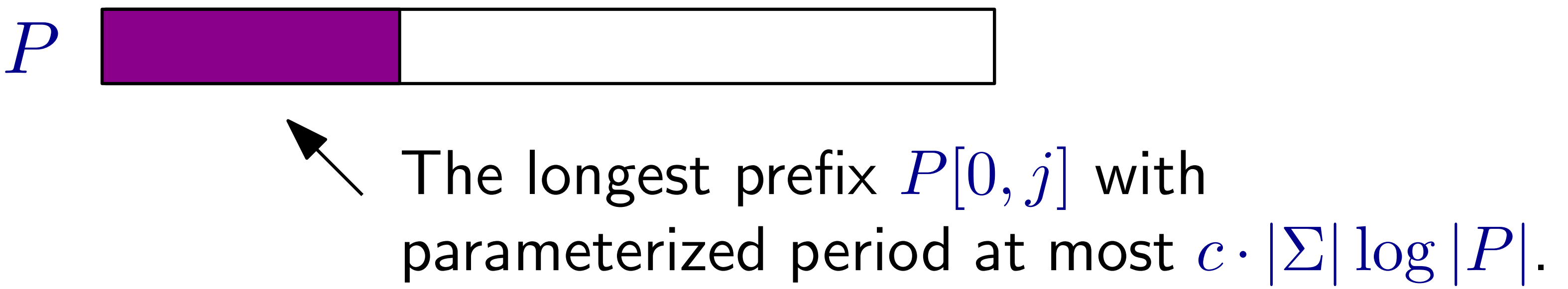

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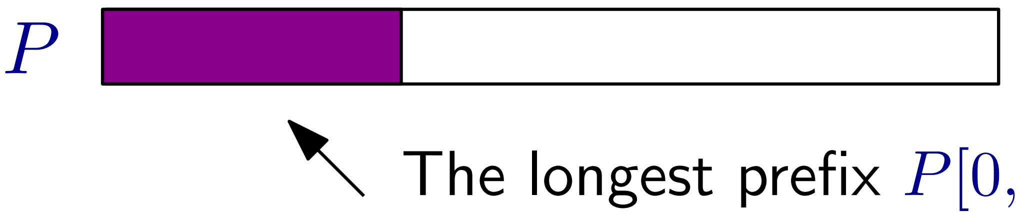

The longest prefix $P[0, j]$ with parameterized period at most $c \cdot |\Sigma| \log |P|$.

Matches with $P[0, j+1]$ are $\Omega(|\Sigma| \log |P|)$ alignments apart.

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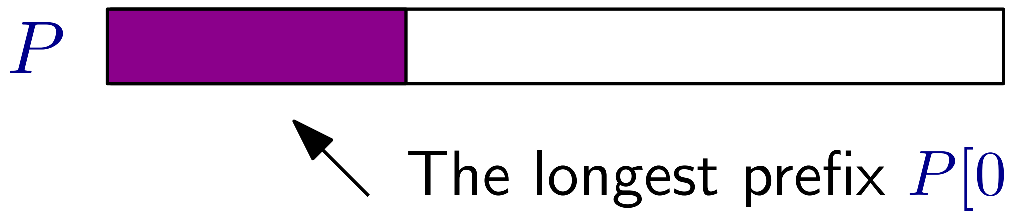

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We also give a deterministic algorithm which outputs all $P[0, j]$ matches in $O(|\Sigma| \log |P|)$ space and $O(1)$ time per character.

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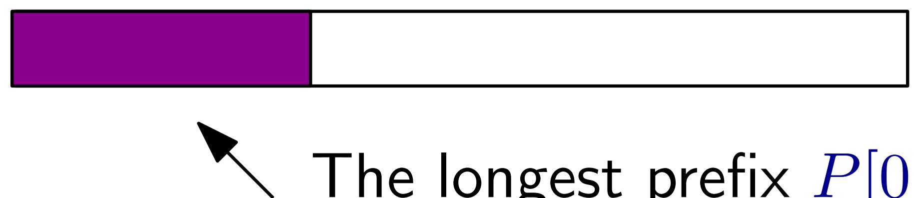
P



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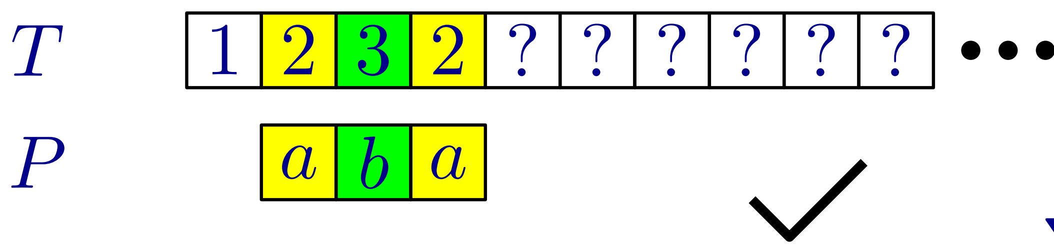
(in fact it works for any pattern with small parameterized period)

Conclusions



Σ is the alphabet

Our Main Results

Parameterized matching can be solved in $O(|\Sigma| \log |P|)$ space and:

- $O(1)$ time per character when $|\Sigma| = \{1, 2, 3, \ldots, |\Sigma|\}$
- $O(\sqrt{\log |\Sigma| / \log \log |\Sigma|})$ time per character for general Σ

Both algorithms are randomised (Monte-Carlo)

Thank you for listening