Lower Bounds for Online Integer Multiplication and Convolution in the Cell-Probe Model

ICALP, 4–8 July 2011

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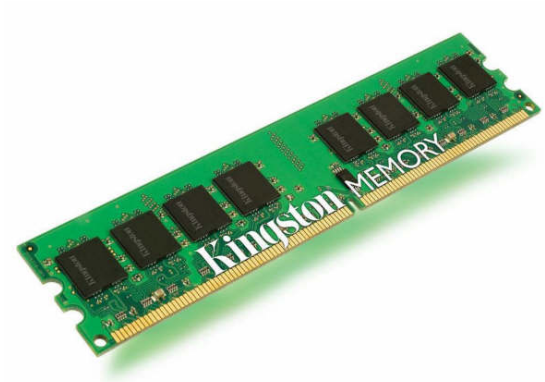
Joint work with

Raphaël Clifford

University of BRISTOL

Cell-probe model

The CPU does not remember anything in between operations.



Cell probes

Read

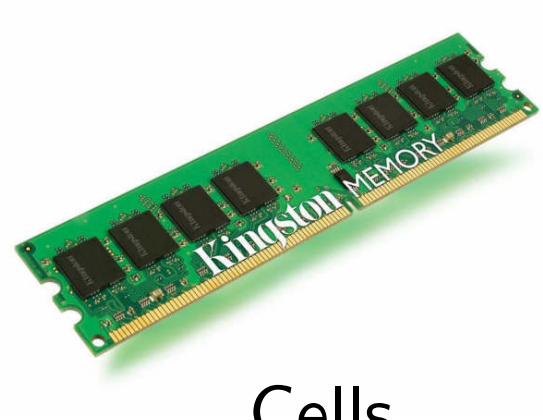
Write

Cells

0
1
2
3
4
5
. . .

w bits

Cell-probe model



The CPU does not remember anything in between operations.

Cell probes

Read

Write

Cells

0
1
2
3
4
5
...

w bits

The CPU has unlimited computational power.

Multiplication (problem 1)

$\times$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$\begin{array}{rr} & 2 \\ \times & 3 \\ \hline \end{array}$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$\begin{array}{r} 2 \\ \times \quad 3 \\ \hline 6 \end{array}$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$\begin{array}{rr} & 4\ 2 \\ \times & 8\ 3 \\ \hline & 6 \end{array}$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$\begin{array}{rr} & 4\ 2 \\ \times & 8\ 3 \\ \hline & 8\ 6 \end{array}$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$
\begin{array}{rr}
 & 0\ 4\ 2 \\
\times & 9\ 8\ 3 \\
\hline
 & 2\ 8\ 6
\end{array}
$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$
\begin{array}{rr}
 & 1\ 0\ 4\ 2 \\
\times & 2\ 9\ 8\ 3 \\
\hline
 & 8\ 2\ 8\ 6
\end{array}
$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$
\begin{array}{rr}
 & 4\ 1\ 0\ 4\ 2 \\
\times & 4\ 2\ 9\ 8\ 3 \\
\hline
 & 0\ 8\ 2\ 8\ 6
\end{array}
$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$
\begin{array}{rr}
 & 6\ 4\ 1\ 0\ 4\ 2 \\
\times & 0\ 4\ 2\ 9\ 8\ 3 \\
\hline
 & 9\ 0\ 8\ 2\ 8\ 6
\end{array}
$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$
\begin{array}{rr}
 & 2\ 6\ 4\ 1\ 0\ 4\ 2 \\
\times & 1\ 0\ 4\ 2\ 9\ 8\ 3 \\
\hline
 & 1\ 9\ 0\ 8\ 2\ 8\ 6
\end{array}
$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$
\begin{array}{rc}
 & \overbrace{}^{n} \\
 & 5\;2\;6\;4\;1\;0\;4\;2 \\
\times & 7\;1\;0\;4\;2\;9\;8\;3 \\
\hline
 & 5\;1\;9\;0\;8\;2\;8\;6
\end{array}
$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$
\begin{array}{rcccccccccccccccc}
 & & & & & & & & & \overbrace{}^{n} & & & & & & \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 2 & 6 & 4 & 1 & 0 & 4 & 2 \\
\times & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 0 & 4 & 2 & 9 & 8 & 3 \\
\hline
 & 3 & 7 & 3 & 9 & 7 & 7 & 6 & 6 & 5 & 1 & 9 & 0 & 8 & 2 & 8 & 6
\end{array}
$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$
\begin{array}{rc}
 & \overbrace{}^{n} \\
 & 5\;2\;6\;4\;1\;0\;4\;2 \\
\times & 7\;1\;0\;4\;2\;9\;8\;3 \\
\hline
 & 5\;1\;9\;0\;8\;2\;8\;6
\end{array}
$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Multiplication (problem 1)

$$
\begin{array}{rcccccccc}
 & \overbrace{}^{n} & & & & & & & \\
 & 5 & 2 & 6 & 4 & 1 & 0 & 4 & 2 \\
\times & 7 & 1 & 0 & 4 & 2 & 9 & 8 & 3 \\
\hline
 & 5 & 1 & 9 & 0 & 8 & 2 & 8 & 6
\end{array}
$$

Digits from the set $[q]$ (base q)

Notation: $[q] = \{0, \ldots, q-1\}$

Time lower bound: $\Omega\left(\frac{\delta}{w} \cdot n \log n\right)$

$\delta = \log q$

Cell-probe model with word size w bits

Multiplication (problem 1)

M. S. Paterson, M. J. Fischer and A. R. Meyer
An Improved Overlap Argument for On-Line Multiplication
SIAM-AMS Proceedings, 1974

For binary numbers on

- Multitape Turing machine: $\Omega(n \log n)$
- Some "random access machines": $\Omega\left(\frac{n \log n}{\log \log n}\right)$

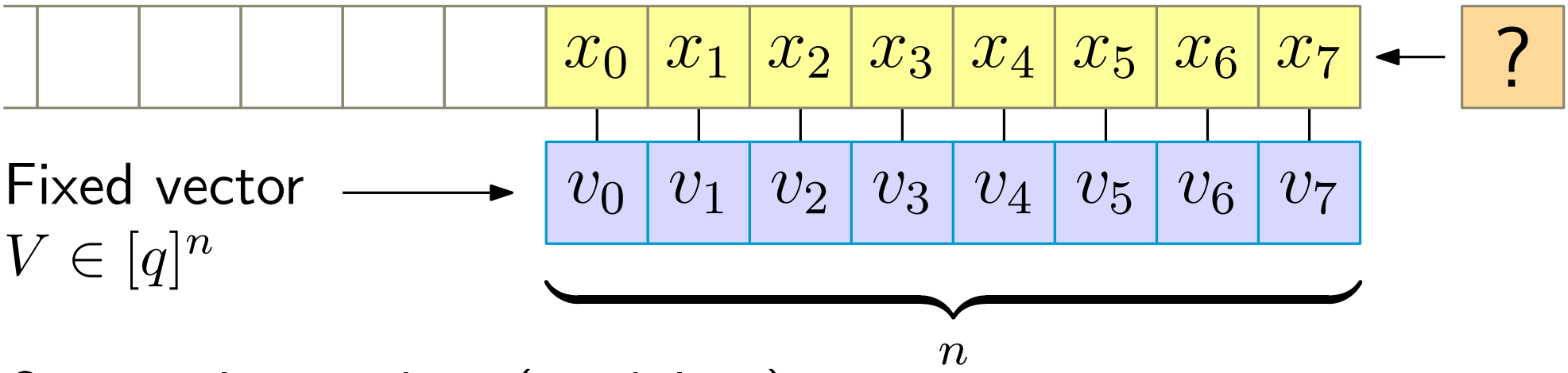
Time lower bound: $\Omega\left(\frac{\delta}{w} \cdot n \log n\right)$

$\delta = \log q$

Cell-probe model with word size w bits

Convolution (problem 2)

Stream of numbers from $[q]$

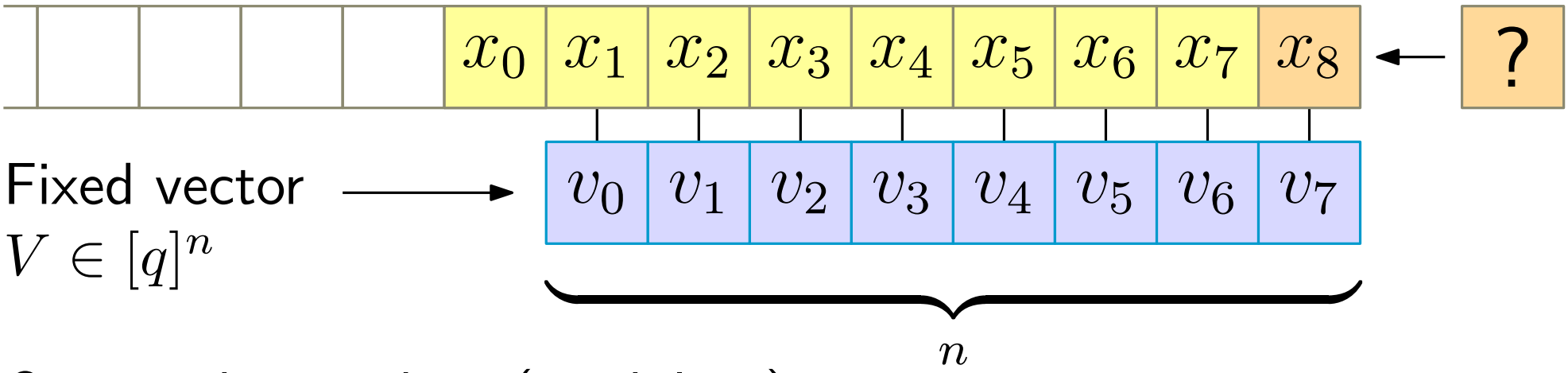


Output dot product (modulo q):

$$V \cdot (\text{last } n \text{ digits of stream}) = \sum_{i=0}^{n-1} v_i x_{(i + \text{leftmost aligned index})}$$

Convolution (problem 2)

Stream of numbers from $[q]$



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Convolution (problem 2)

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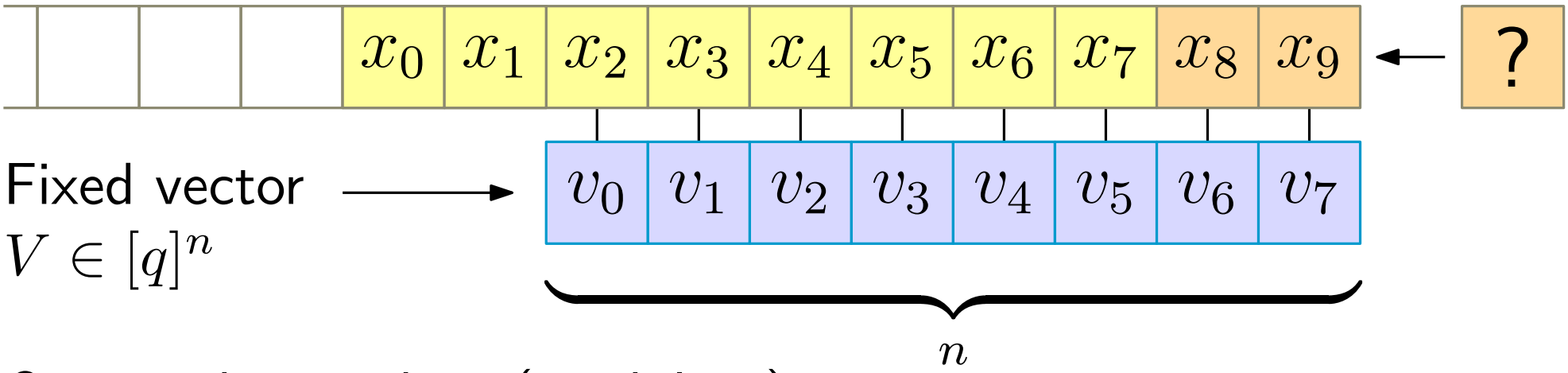

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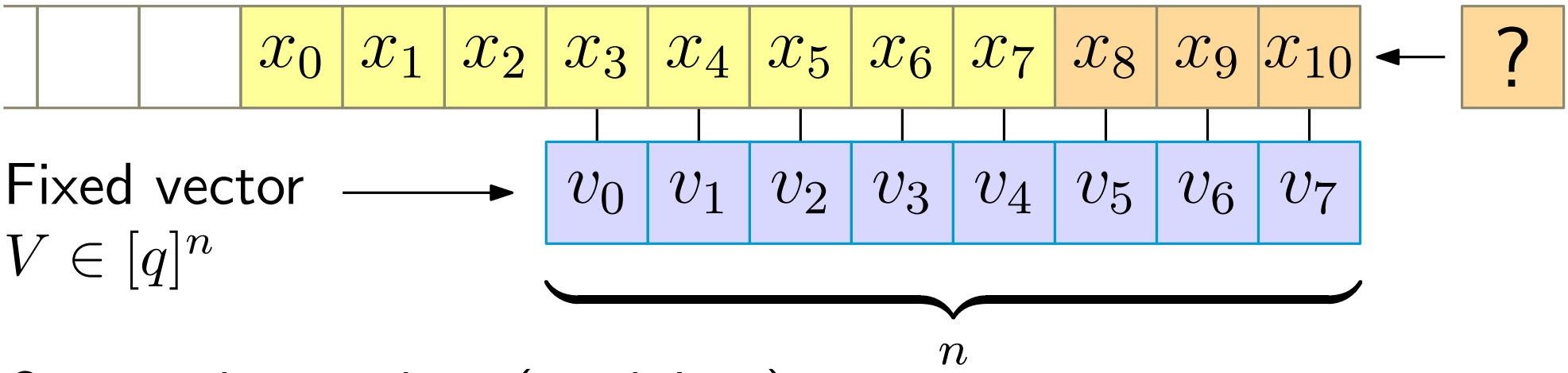
Convolution (problem 2)

Stream of numbers from $[q]$



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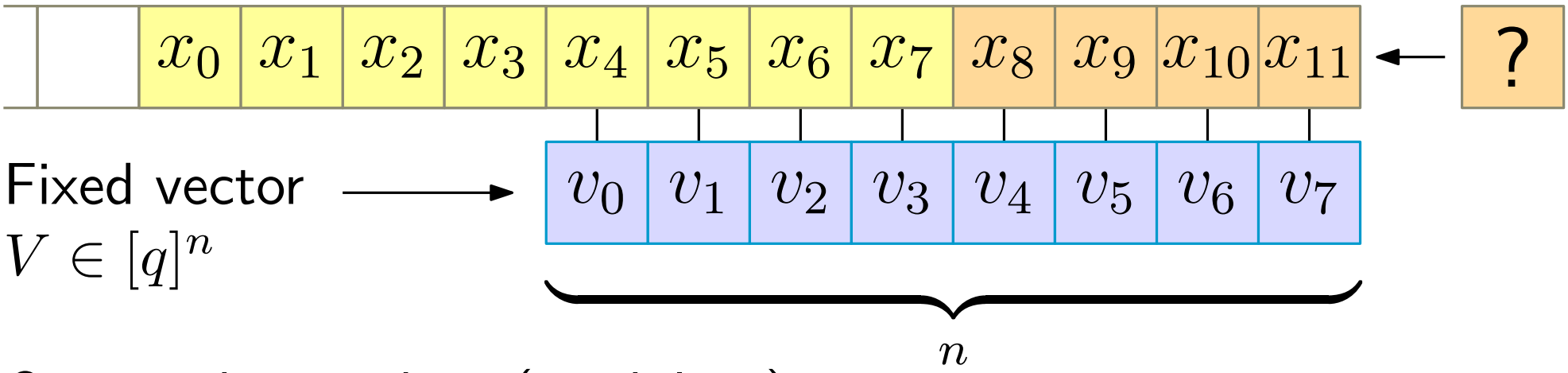
Convolution (problem 2)

Stream of numbers from $[q]$



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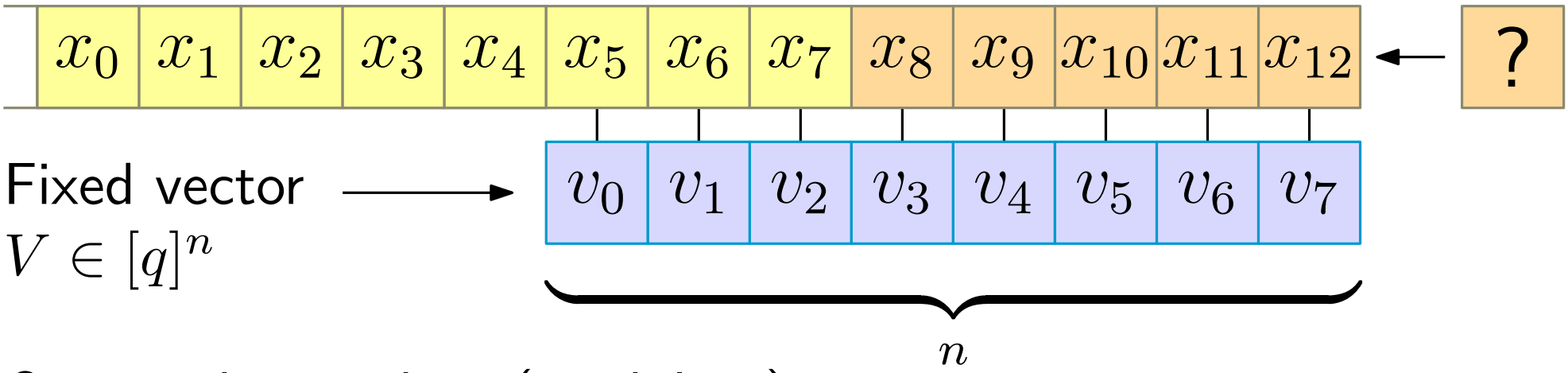
Convolution (problem 2)

Stream of numbers from $[q]$



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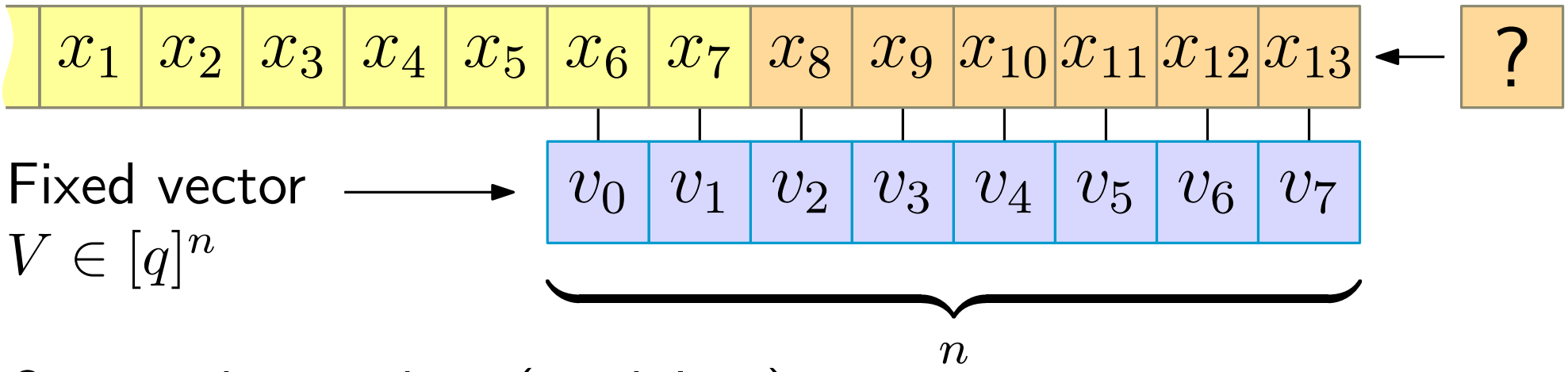
Convolution (problem 2)

Stream of numbers from $[q]$



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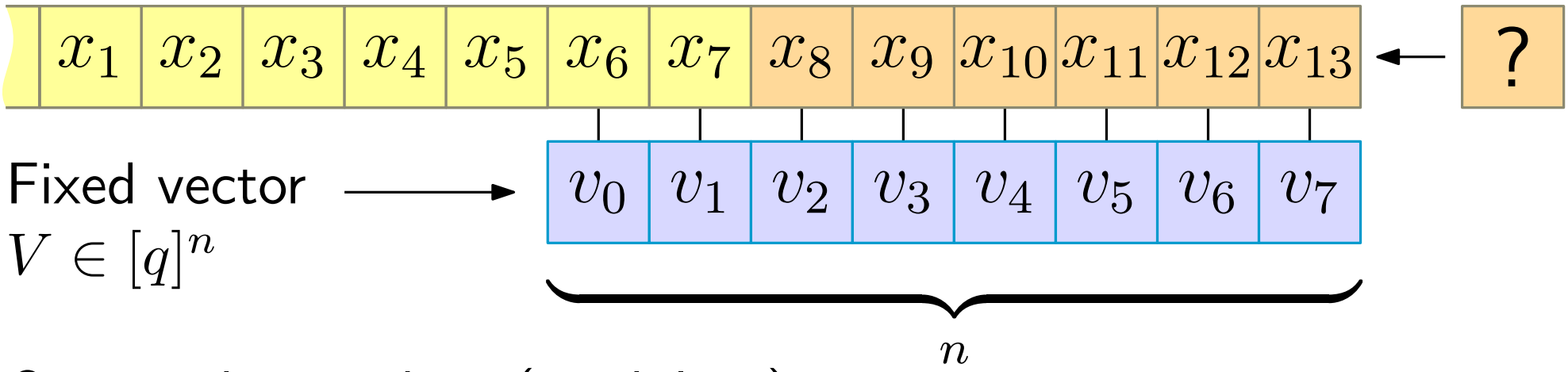
Convolution (problem 2)

Stream of numbers from $[q]$



Output dot product (modulo q):

$$V \cdot (\text{last } n \text{ digits of stream}) = \sum_{i=0}^{n-1} v_i x_{(i + \text{leftmost aligned index})}$$

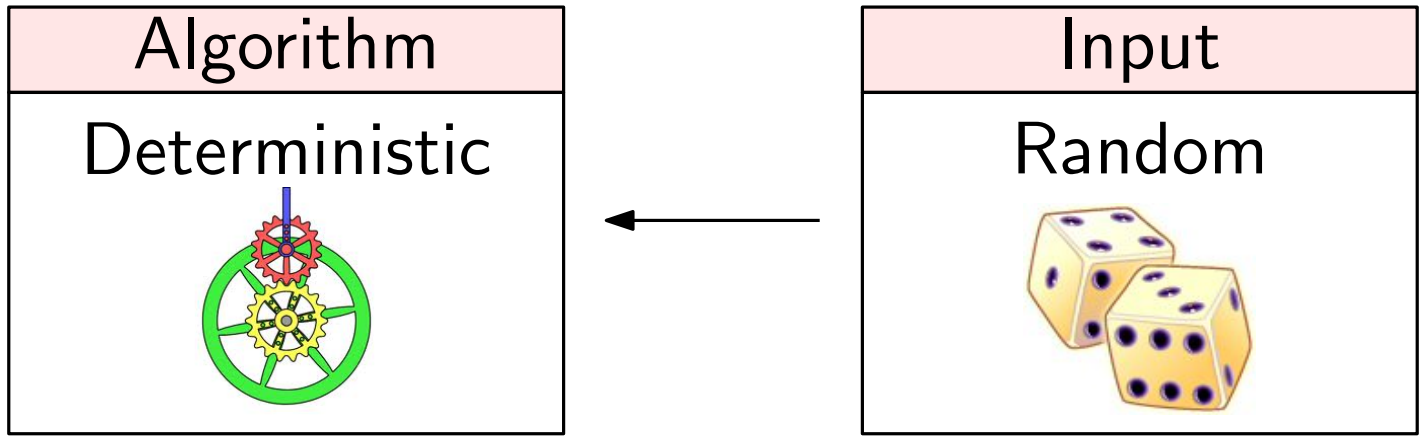
Amortised time lower bound per output: $\Omega\left(\frac{\delta}{w} \log n\right)$

$\delta = \log q$

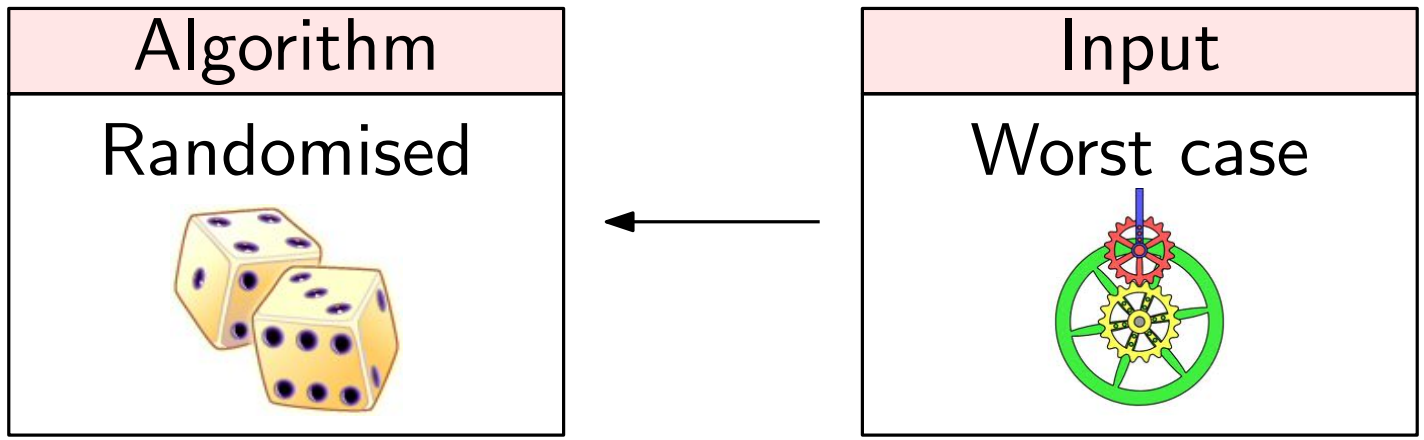
Cell-probe model with word size w bits

Yao's minimax principle

A lower bound on the expected running time for

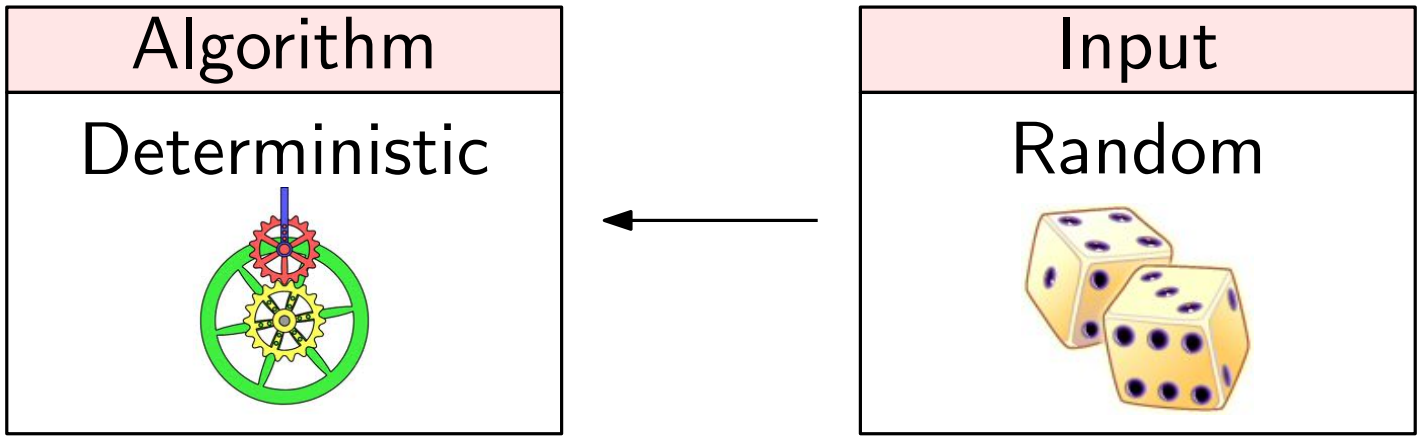


implies that the same lower bound holds for



Yao's minimax principle

A lower bound on the expected running time for



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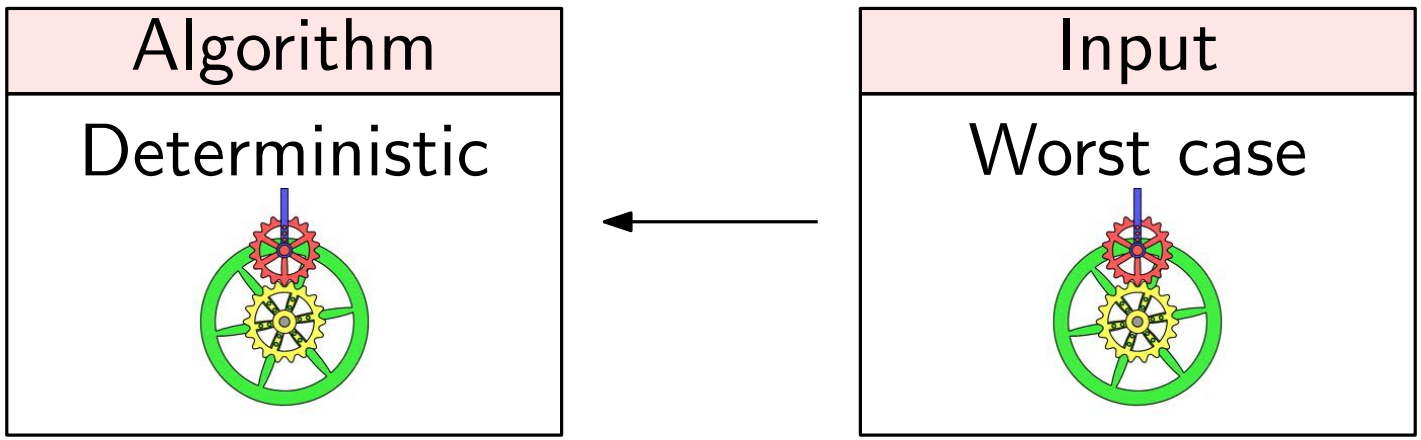
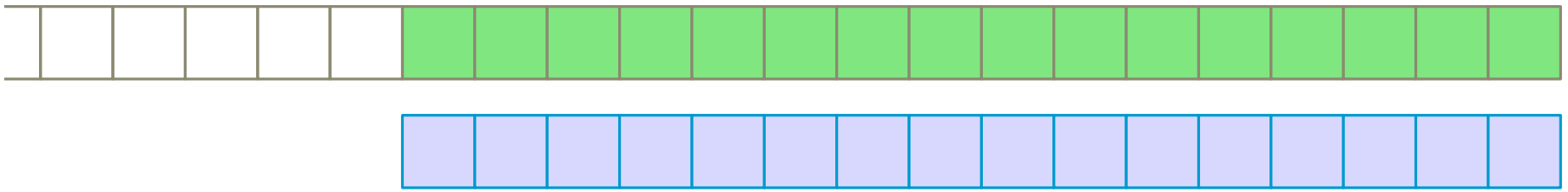


Information transfer

Will use information theoretic arguments from

M. Pătraşcu and E. Demaine
Tight bounds for the partial-sums problem
SODA 2004

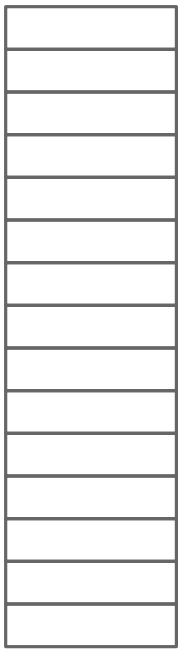
Information transfer



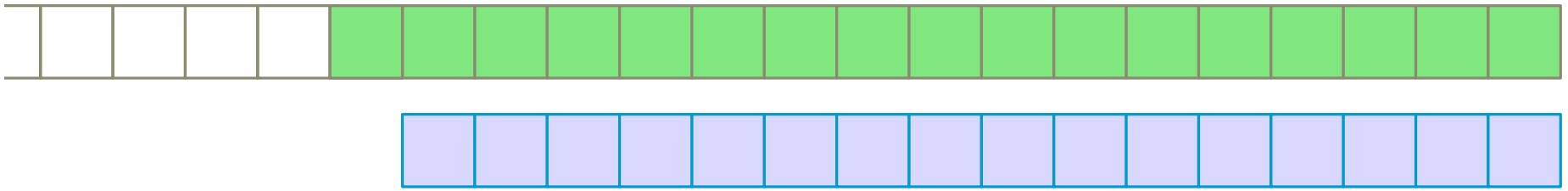
Fixed value

? Unknown value chosen uniformly at random from $[q]$

Memory cells



Information transfer



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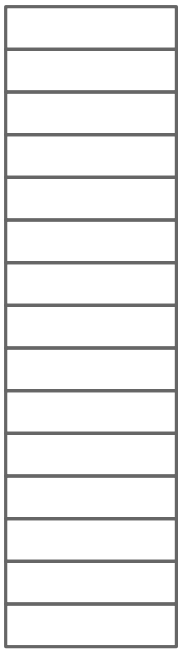
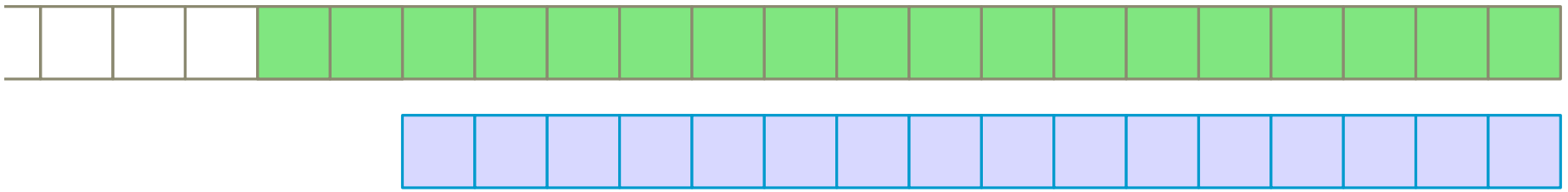
Memory cells



Information transfer



Fixed value

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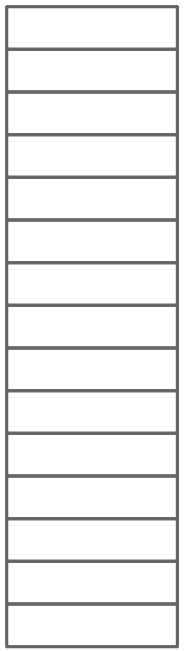
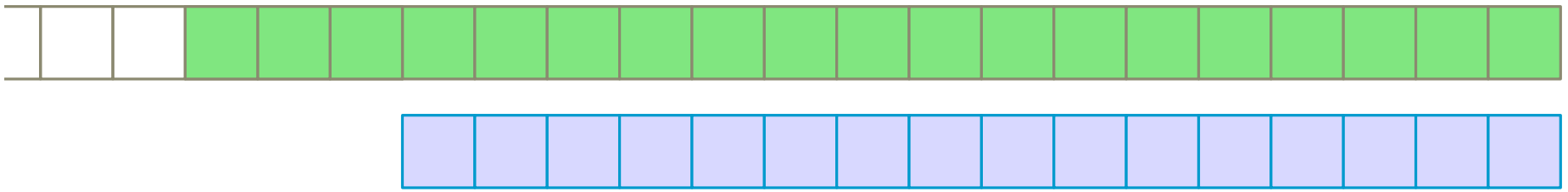


Information transfer



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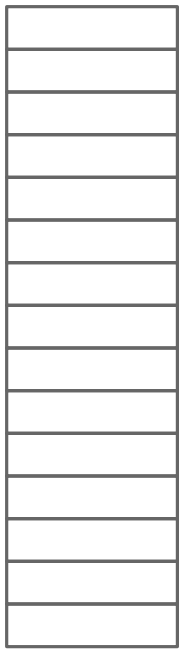
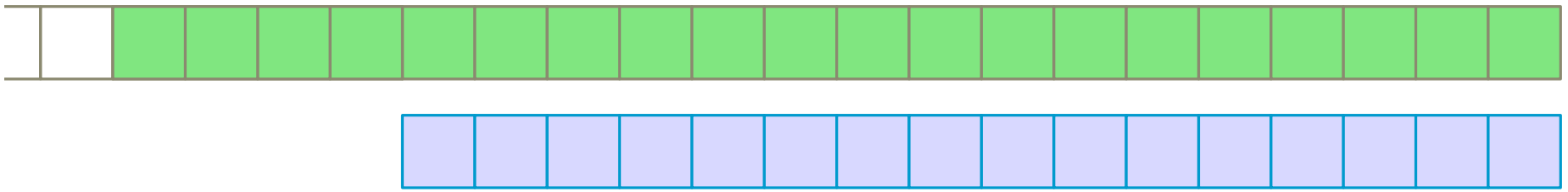


Information transfer



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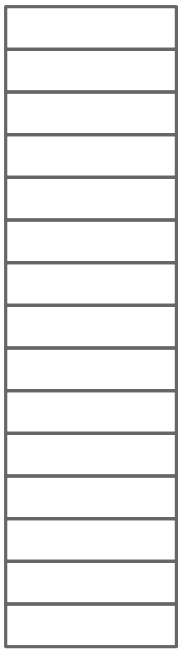
Memory cells



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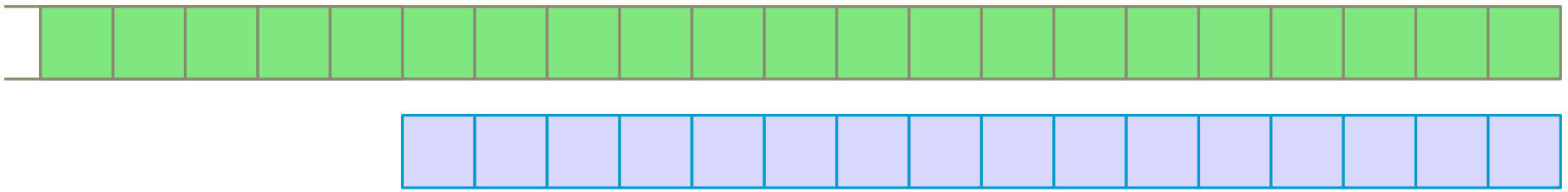

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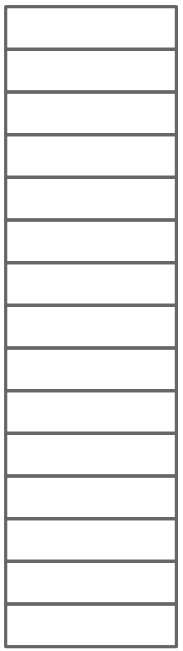
Memory cells



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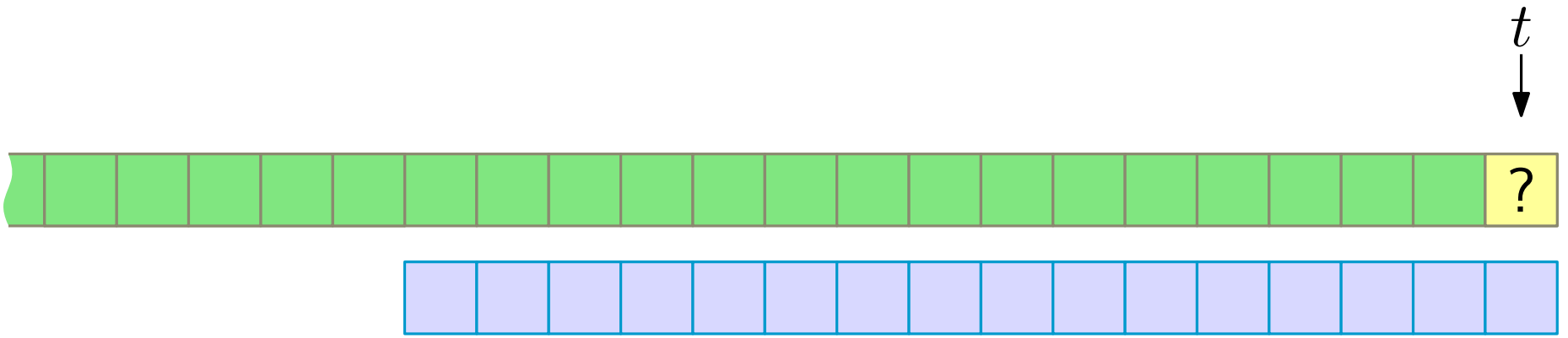

Fixed value

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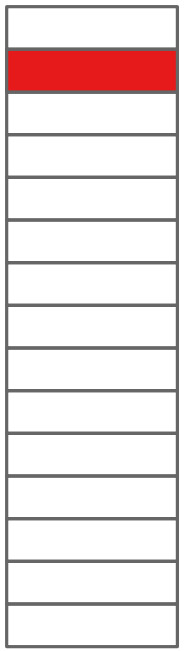
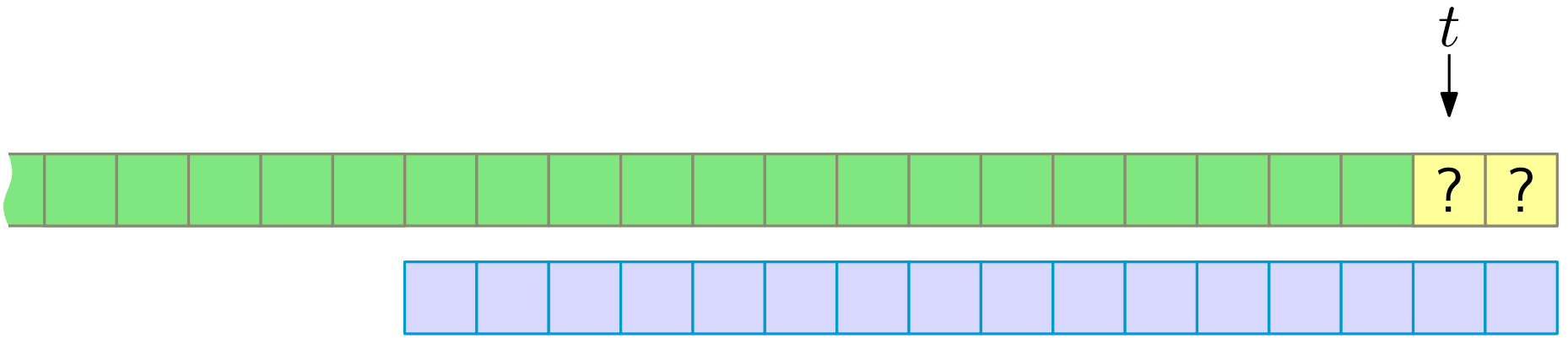
Cell written during the ? -inputs

Memory cells



Information transfer



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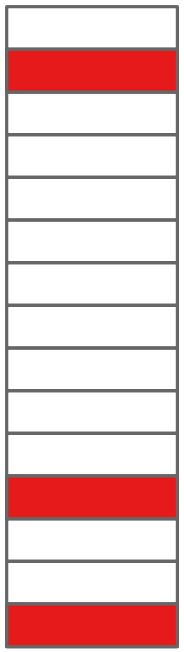
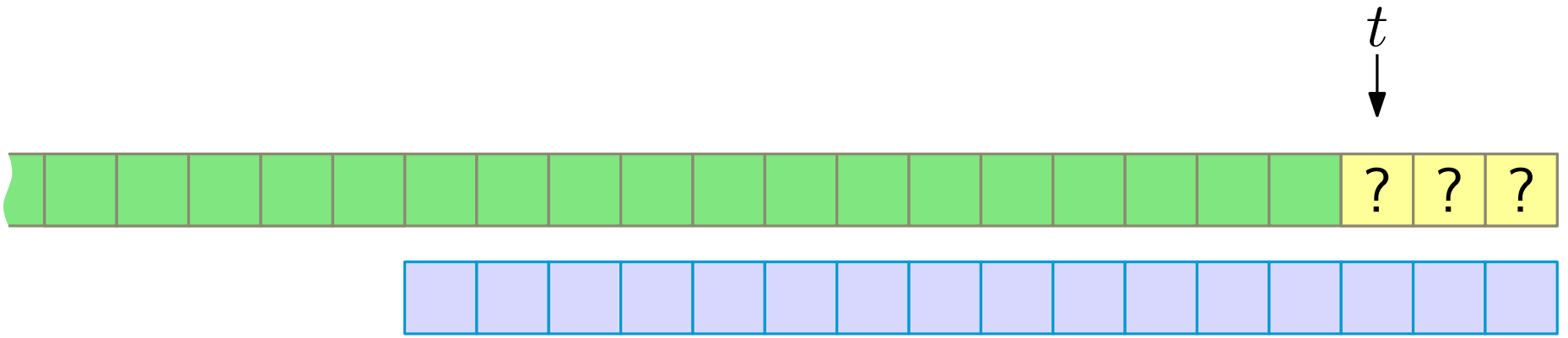


Information transfer



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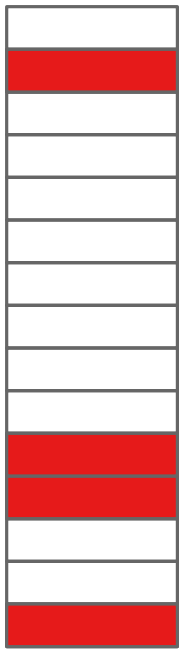
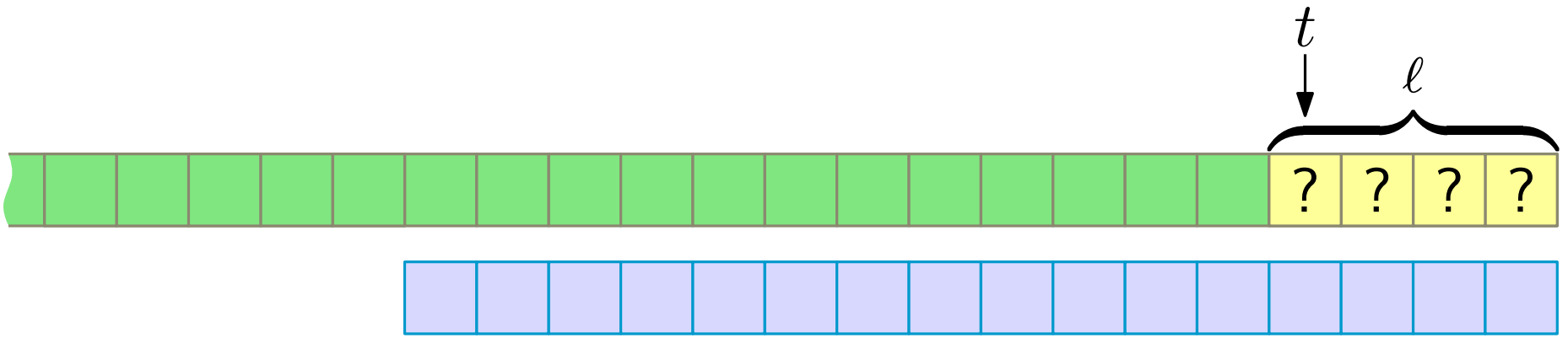


Information transfer



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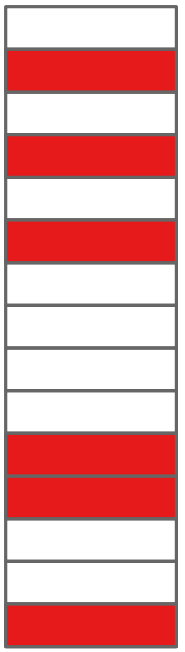
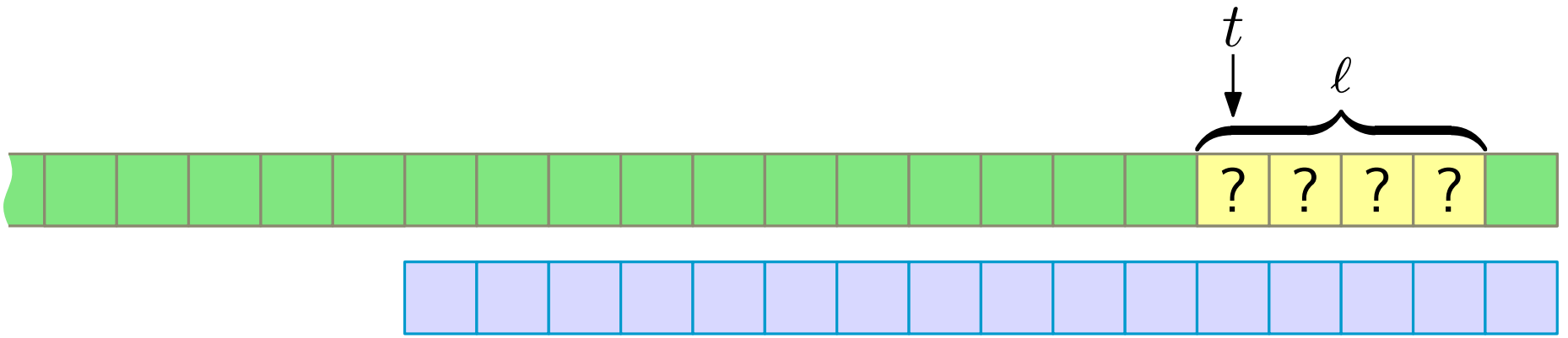


Information transfer



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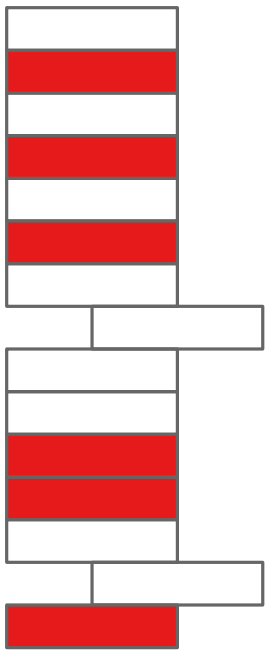
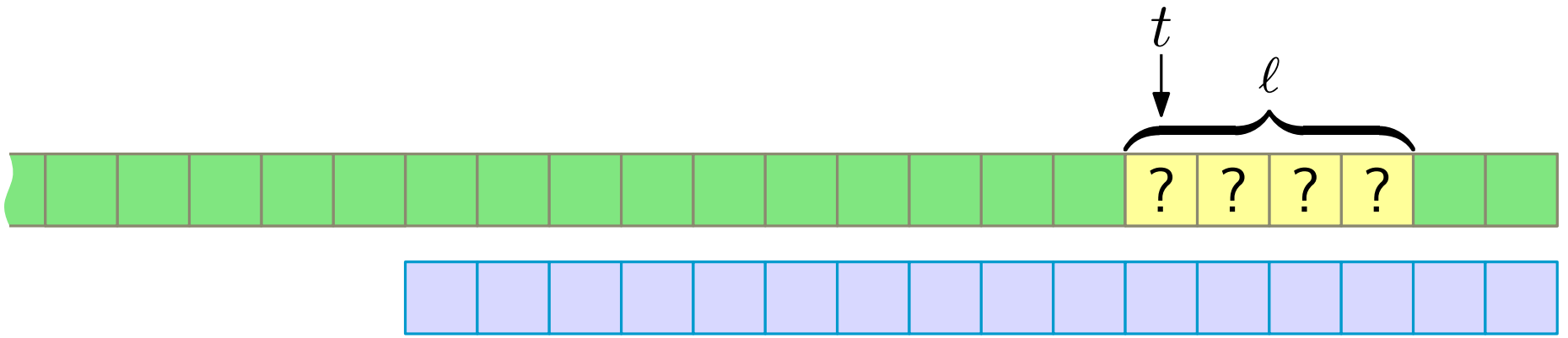
Memory cells



Cell written during the ? -inputs

Cells read during the next ℓ inputs

Information transfer



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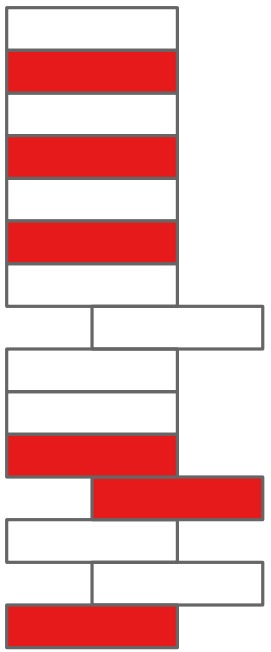

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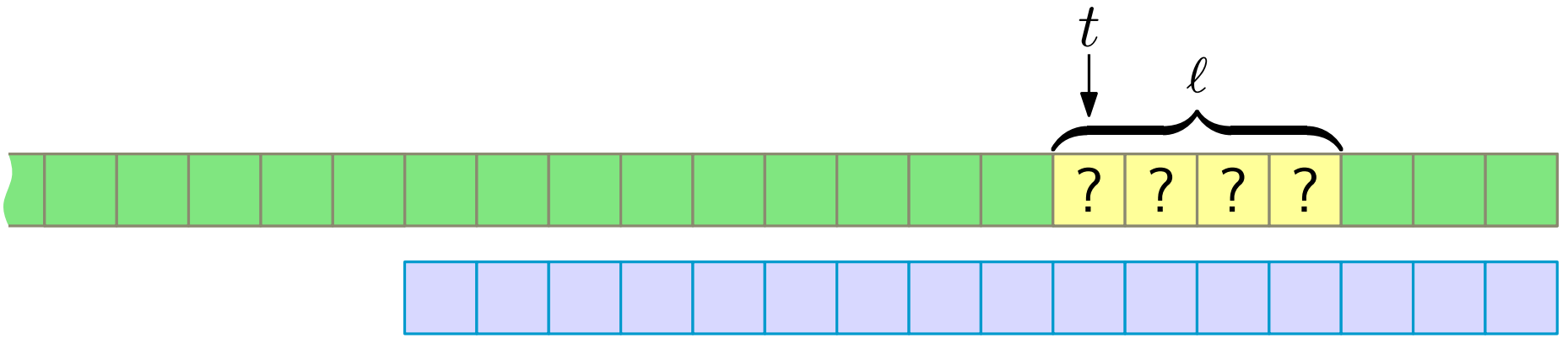
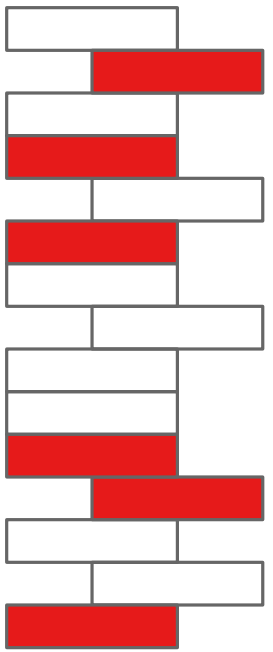


Fixed value

? Unknown value chosen uniformly at random from $[q]$

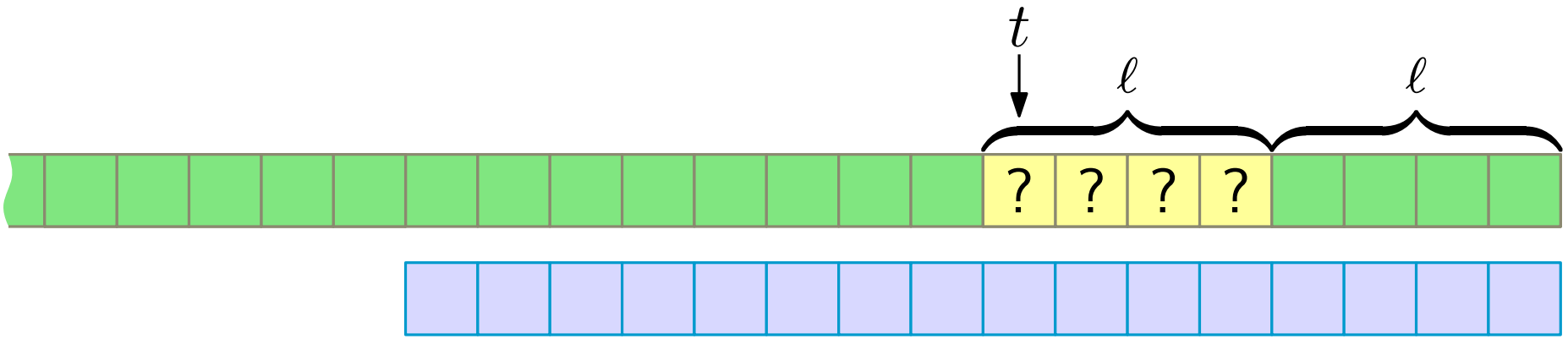
Cell written during the ? -inputs

Memory cells



Cells read during the next ℓ inputs

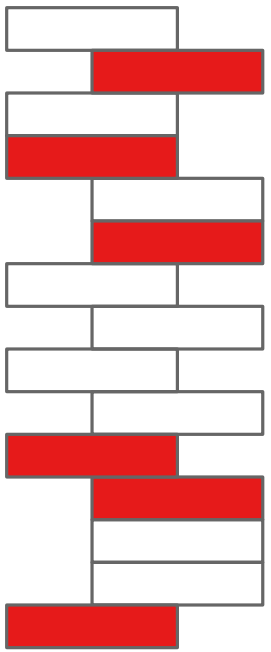
Information transfer



Fixed value

? Unknown value chosen uniformly at random from $[q]$

Memory cells



Cell written during the ? -inputs

Cells read during the next ℓ inputs

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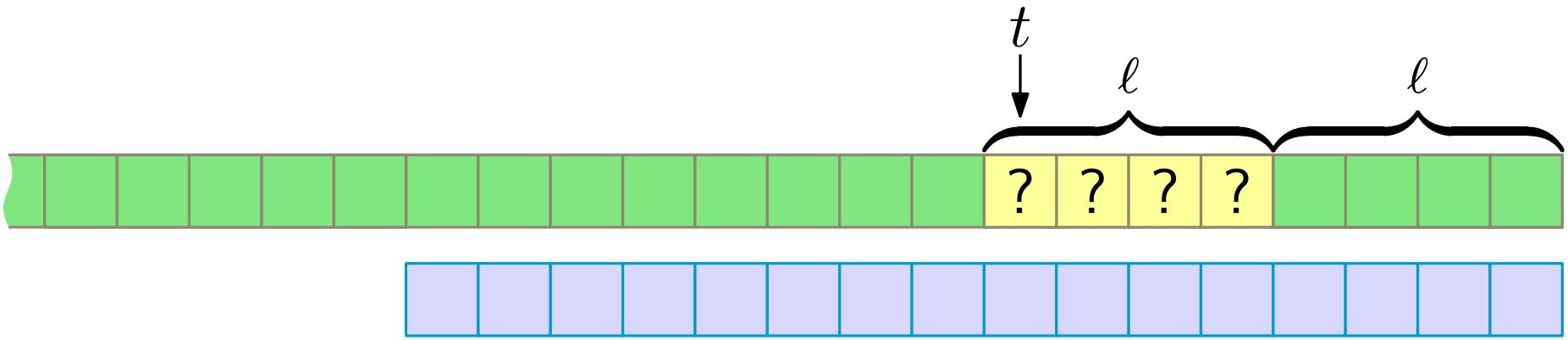
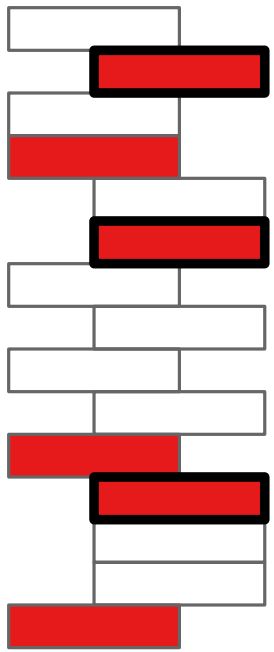


Fixed value

? Unknown value chosen uniformly at random from $[q]$

Memory cells



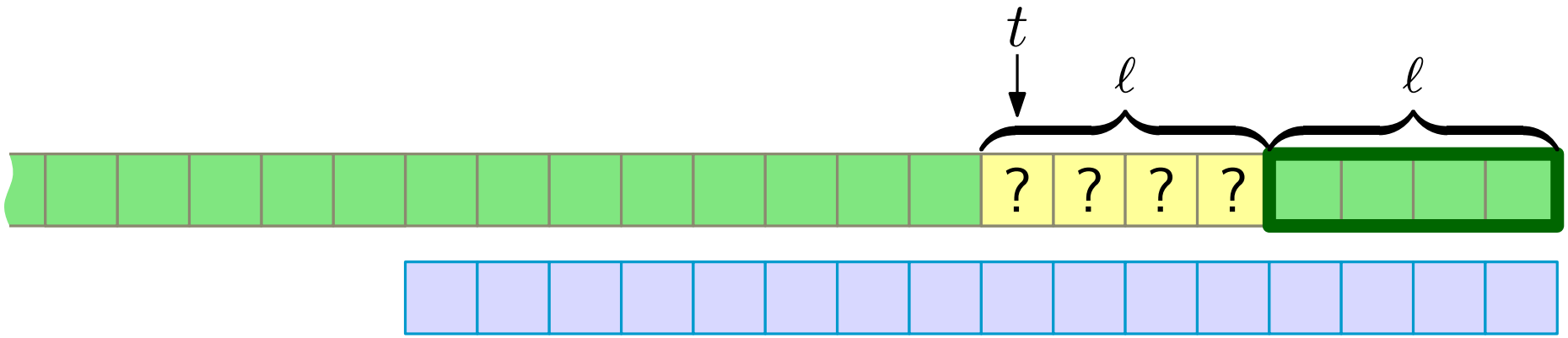
Information transfer $IT(t, \ell)$

Not including cells that were overwritten before being read

Cell written during the ? -inputs

Cells read during the next ℓ inputs

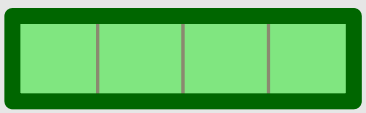
Information transfer



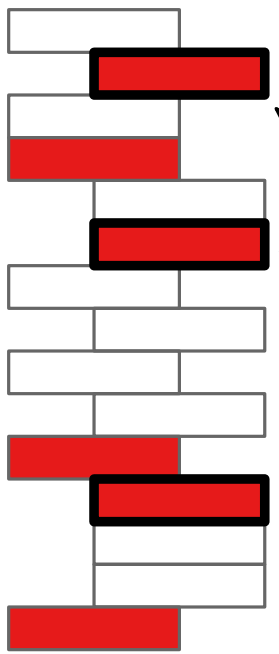
Fixed value

? Unknown value chosen uniformly at random from $[q]$

The cells in $IT(t, \ell)$ provide sufficient information in order to give correct output during



Memory cells

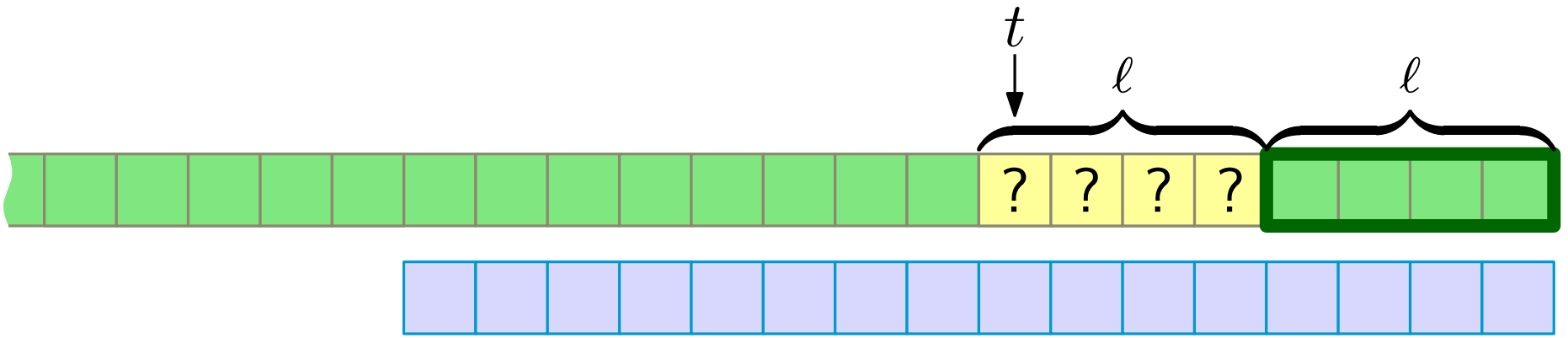


Information transfer $IT(t, \ell)$

Not including cells that were overwritten before being read

inputs

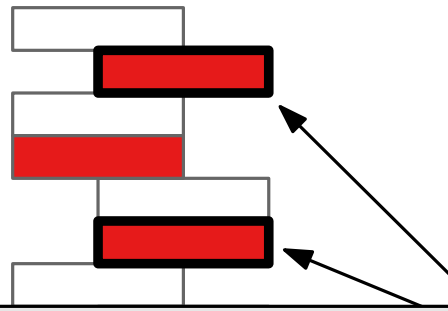
Information transfer



Fixed value

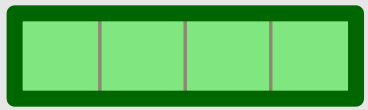
? Unknown value chosen uniformly at random from $[q]$

Memory cells



Information transfer

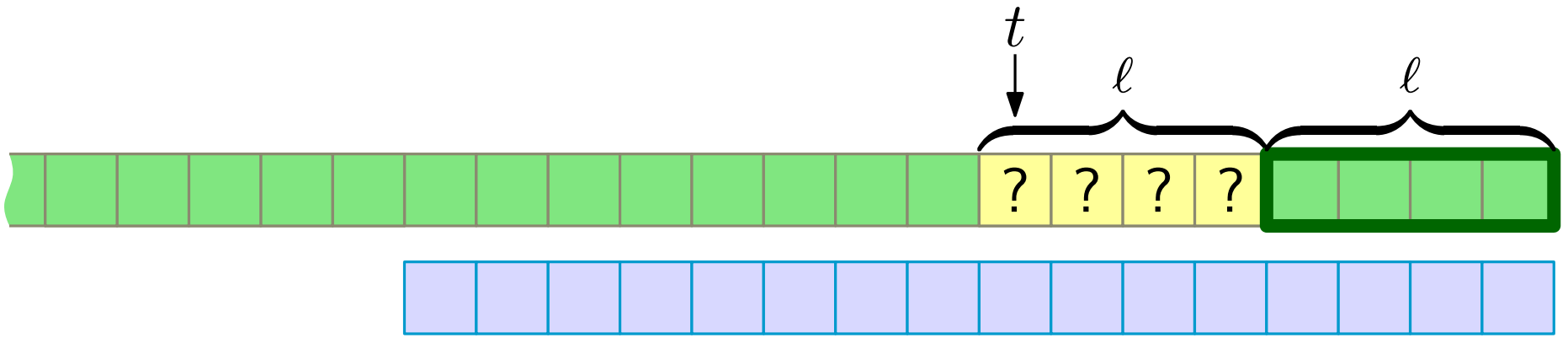
The conditional entropy

$H($the outputs during  | all  fixed$)$

$\leqslant w + 2w \cdot \mathbb{E}\left[|IT(t,\ell)| \ \middle|\ \text{all } \right.$  $\left.\text{ fixed}\right]$

w bits per cell

Information transfer



Fixed value

? Unknown value chosen uniformly at random from $[q]$

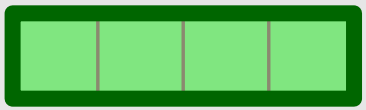
$|IT(t,\ell)|$

Cell	Address	Contents
	00124	76112
	34123	88819
	92540	01882

w bits to encode $|IT(t,\ell)|$

w bits w bits

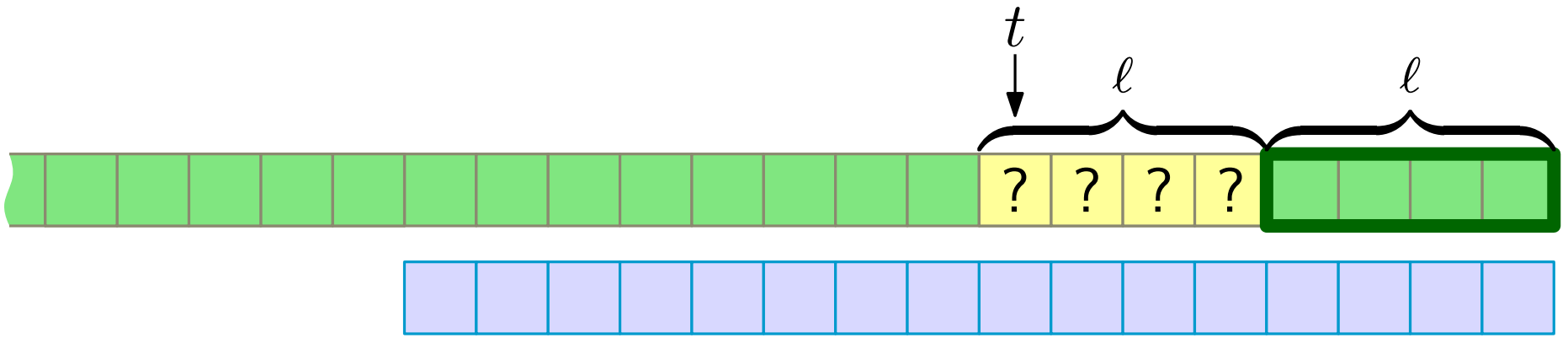
The conditional entropy

$H($the outputs during  | all  fixed$)$

$$\leqslant w + 2w \cdot \mathbb{E}\left[|IT(t,\ell)| \mid \text{all } \square \text{ fixed}\right]$$

w bits per cell

Information transfer



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$|IT(t,\ell)|$

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	00124	76112
	00000	00000
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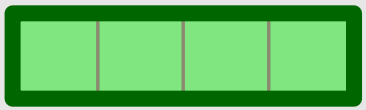
w bits to encode $|IT(t,\ell)|$

w bits w bits

The conditional entropy

$H($the outputs during  | all  fixed$)$

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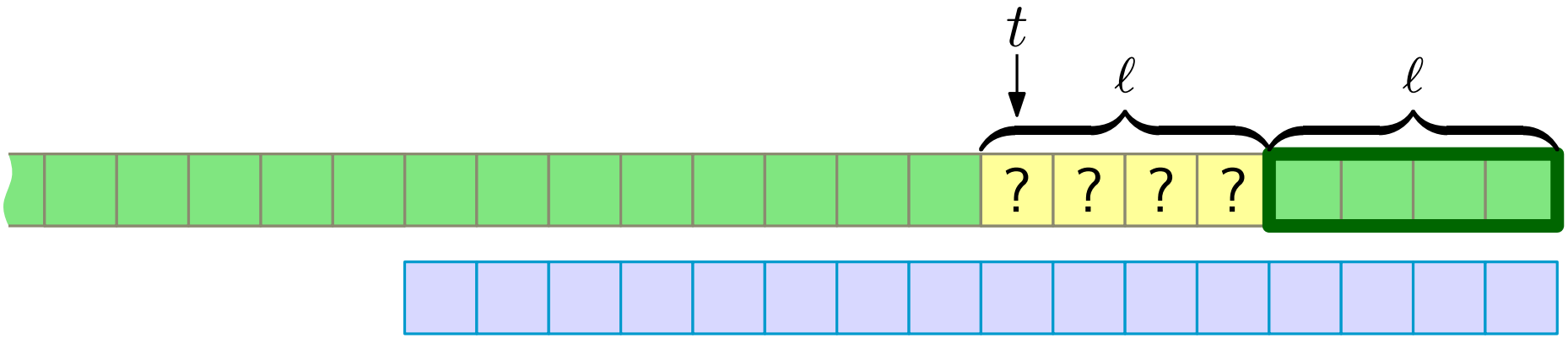
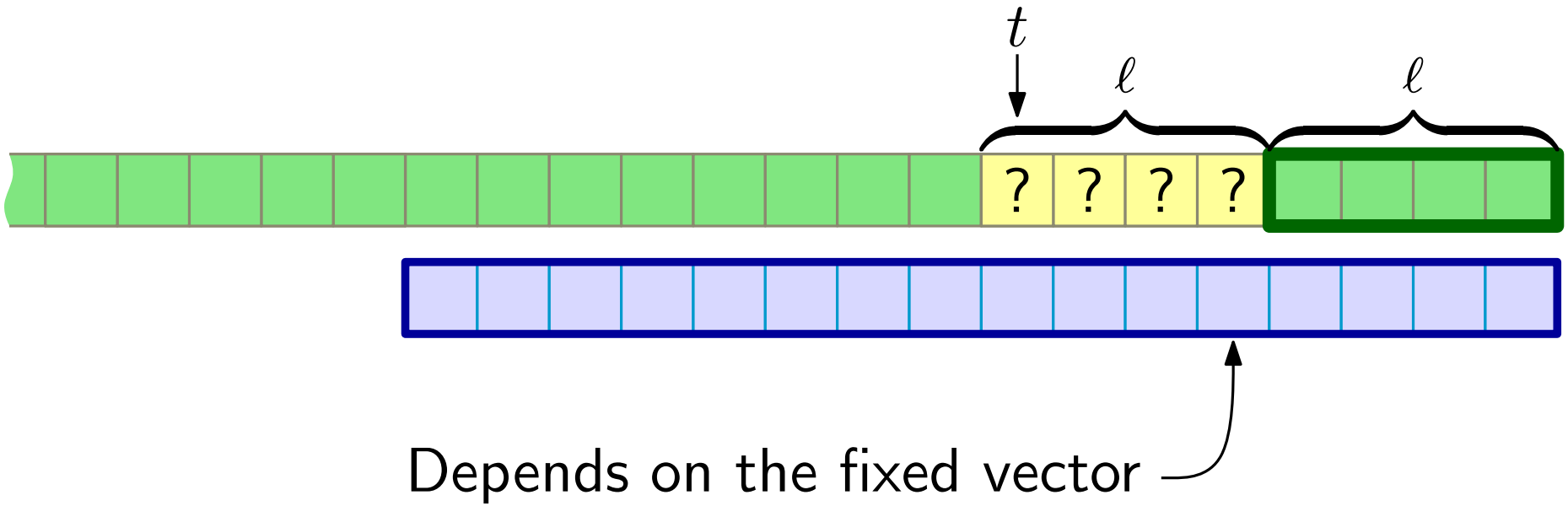
Information transfer



How much information about ? ? ? ? do we **need**

in order to give correct outputs during ?

Information transfer



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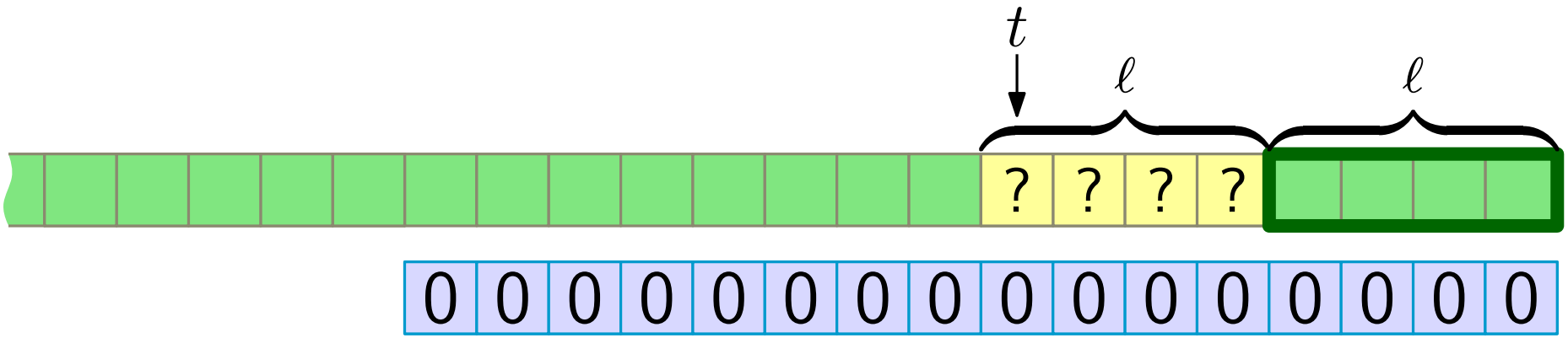
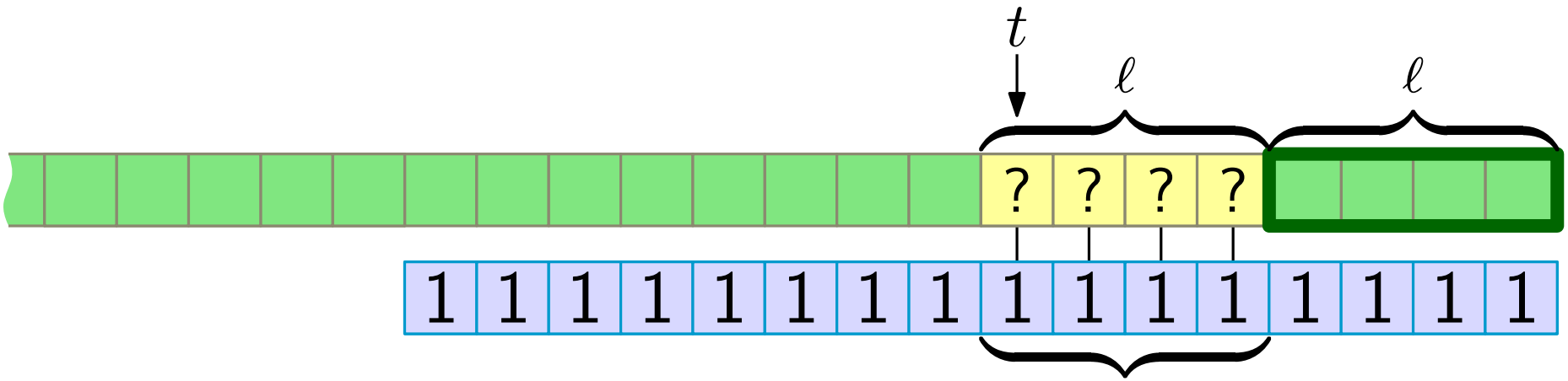


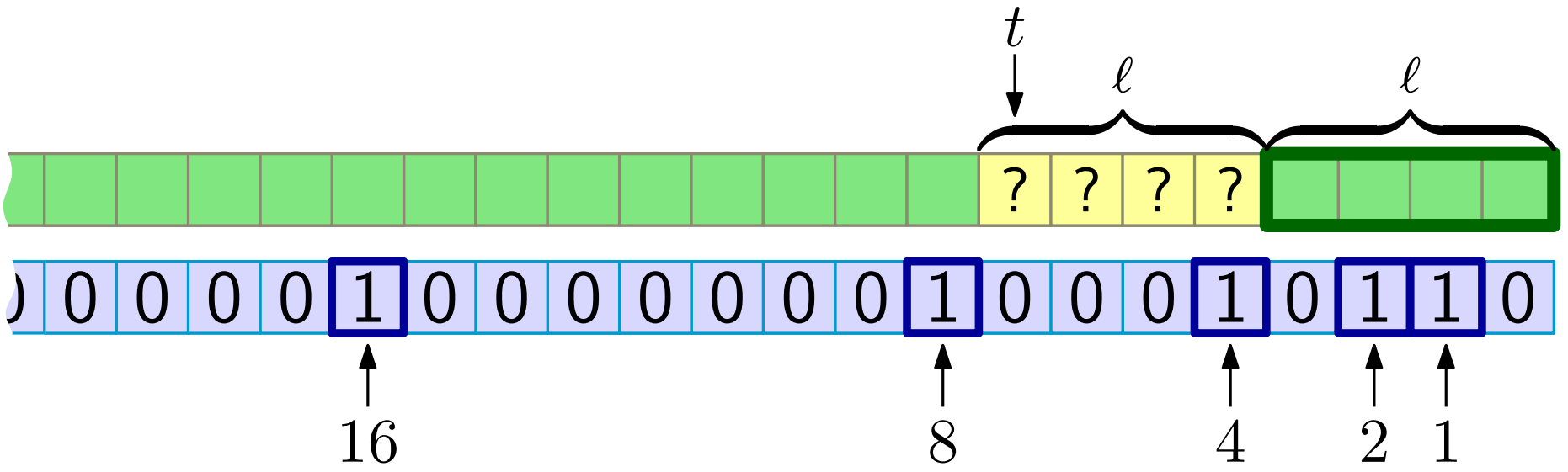
Output is always 0 (no information)

Information transfer



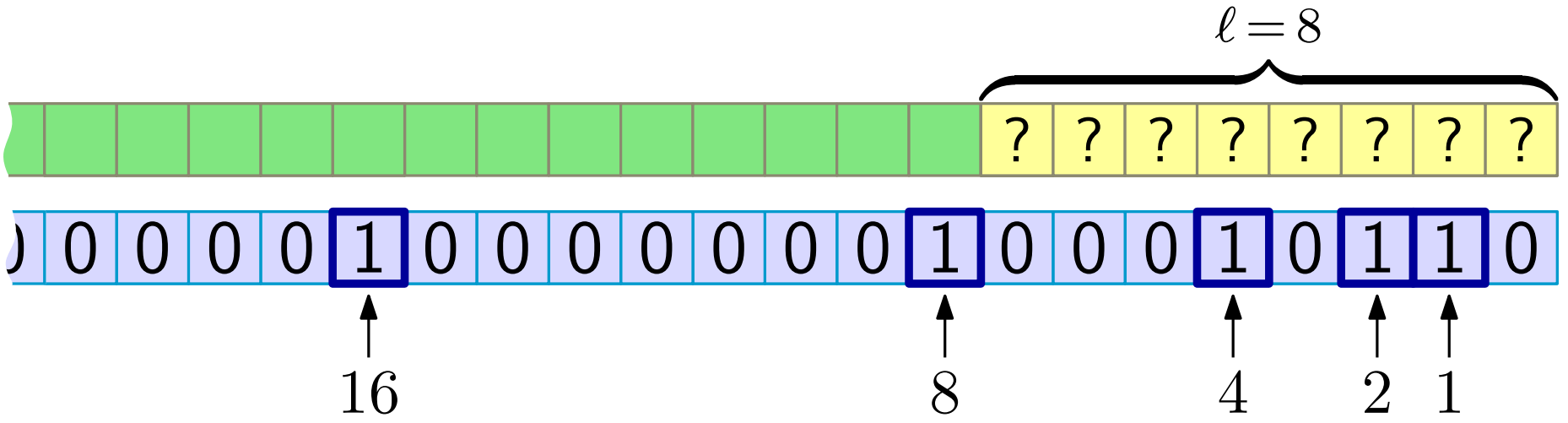
Contributes to the dot product with the same value at each alignment
($\delta = \log q$ bits of information)

Information transfer



1 if the position is a power of 2

Information transfer



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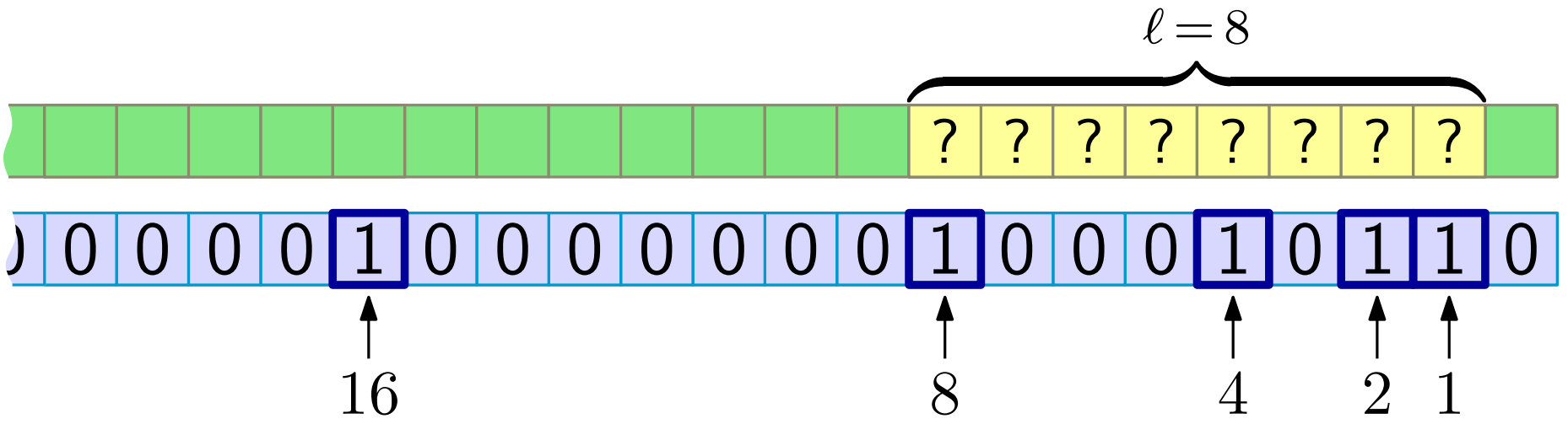

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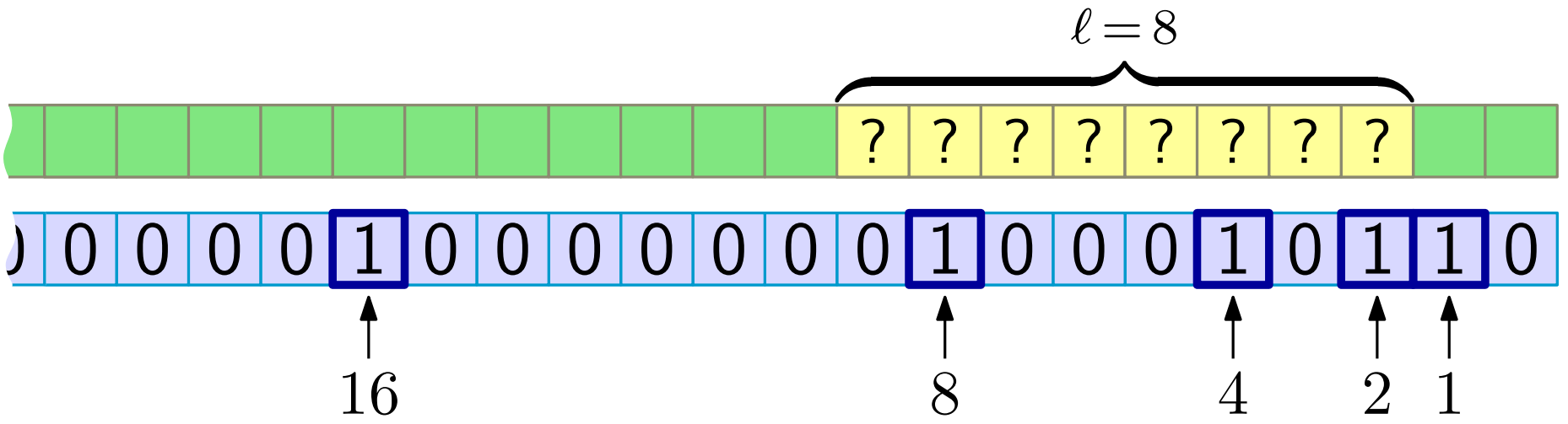
Information transfer



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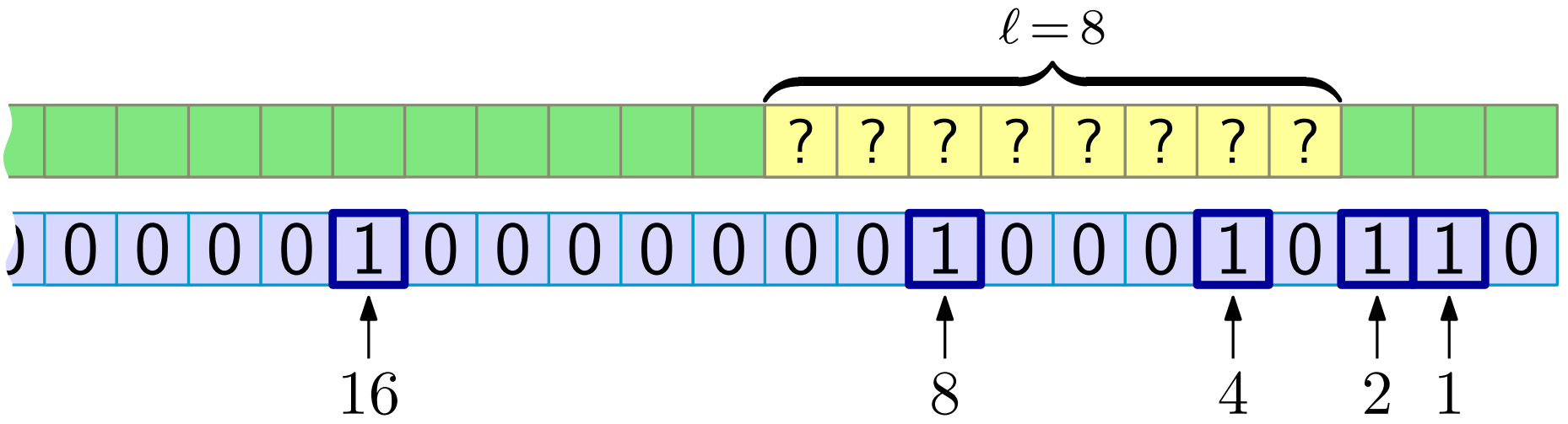

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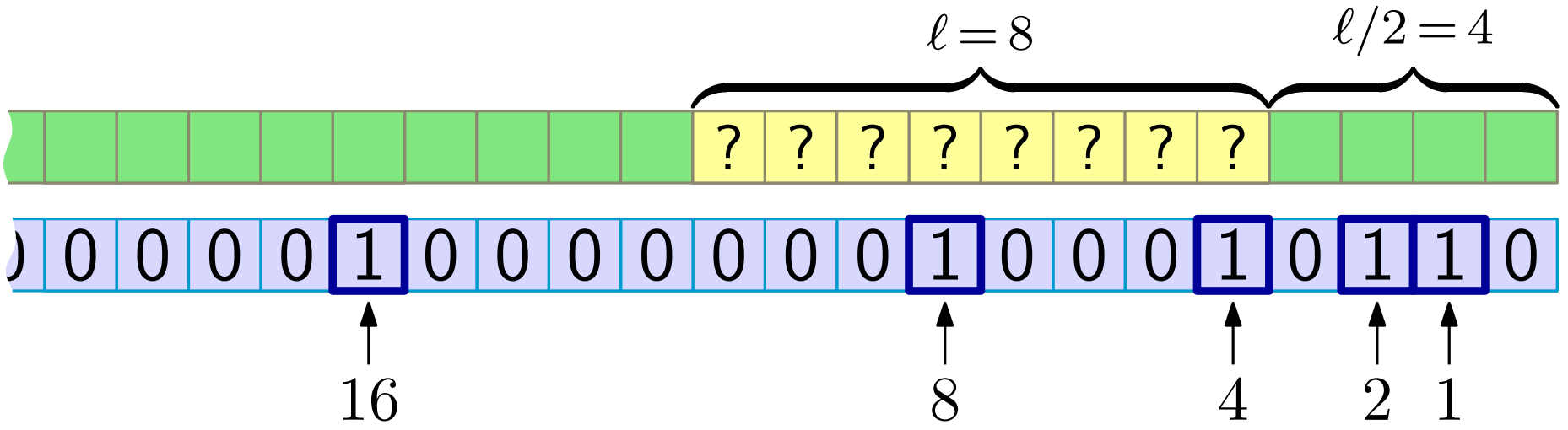

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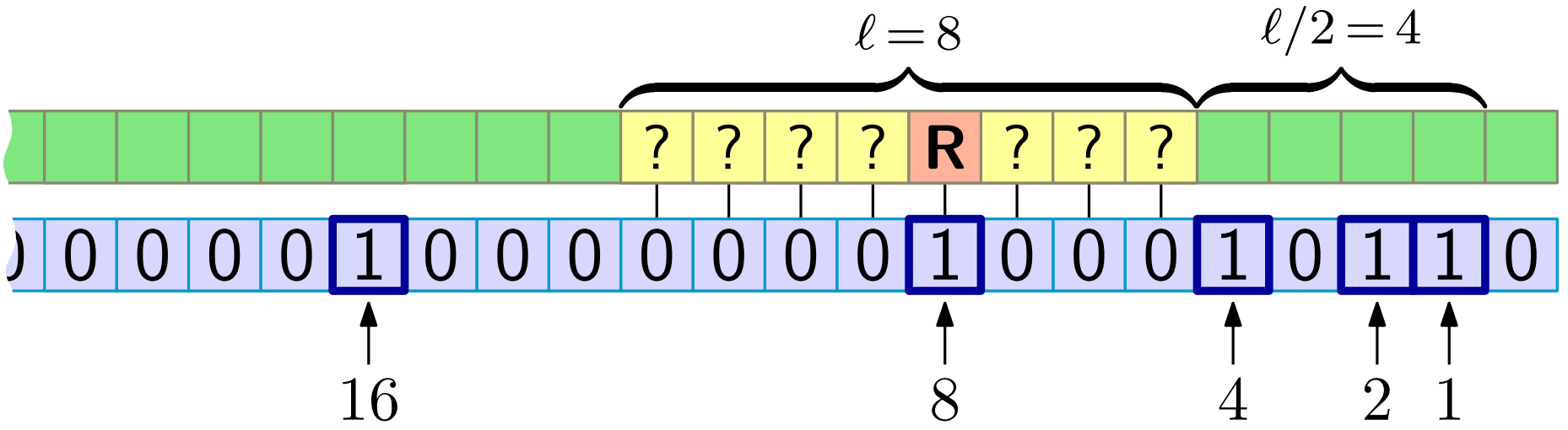
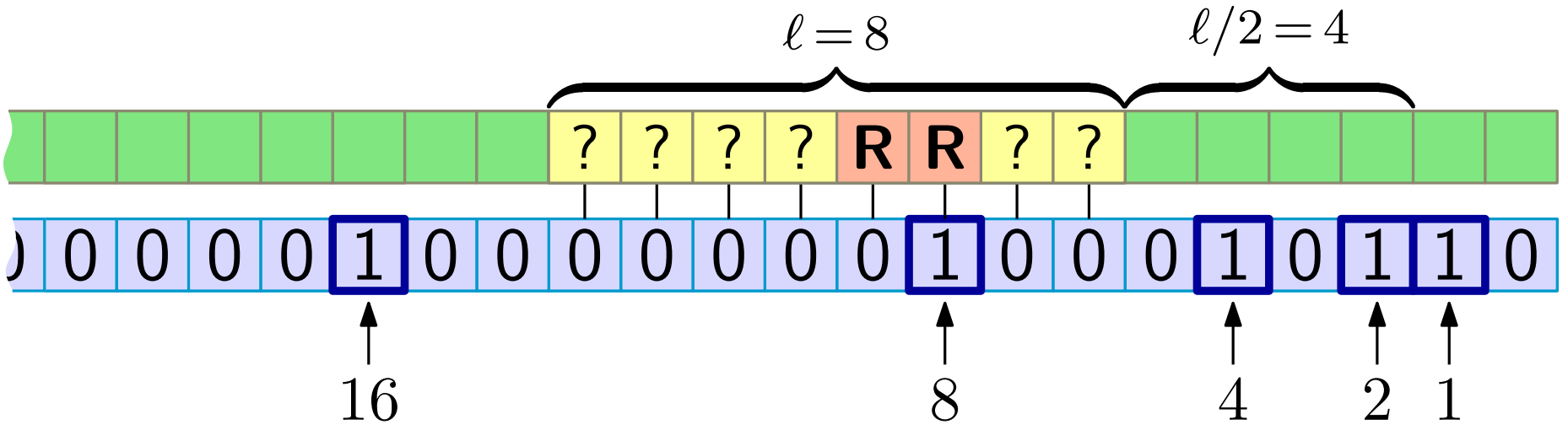


1 if the position is a power of 2

R = a recovered value
(recall that ? is chosen uniformly at random from $[q]$, hence contributes with $\delta = \log q$ bits to the entropy)

Information transfer



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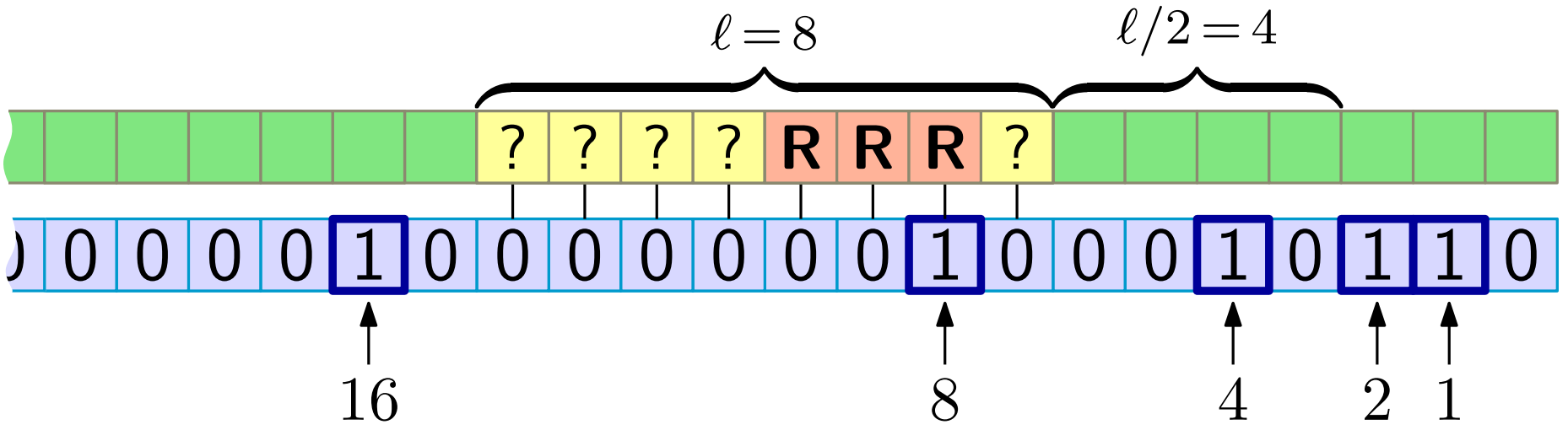

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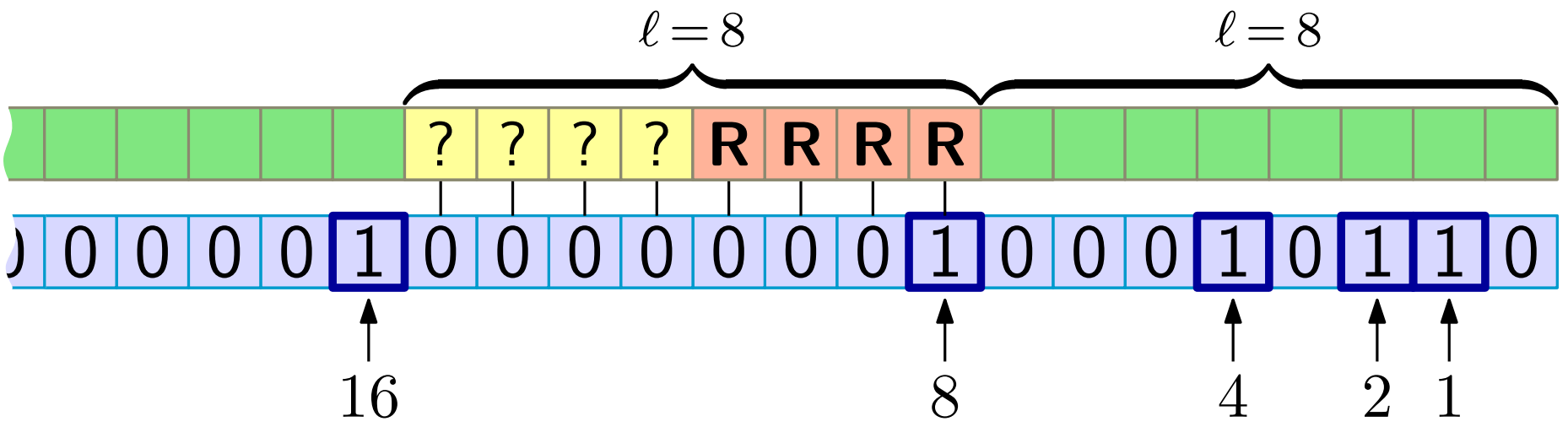

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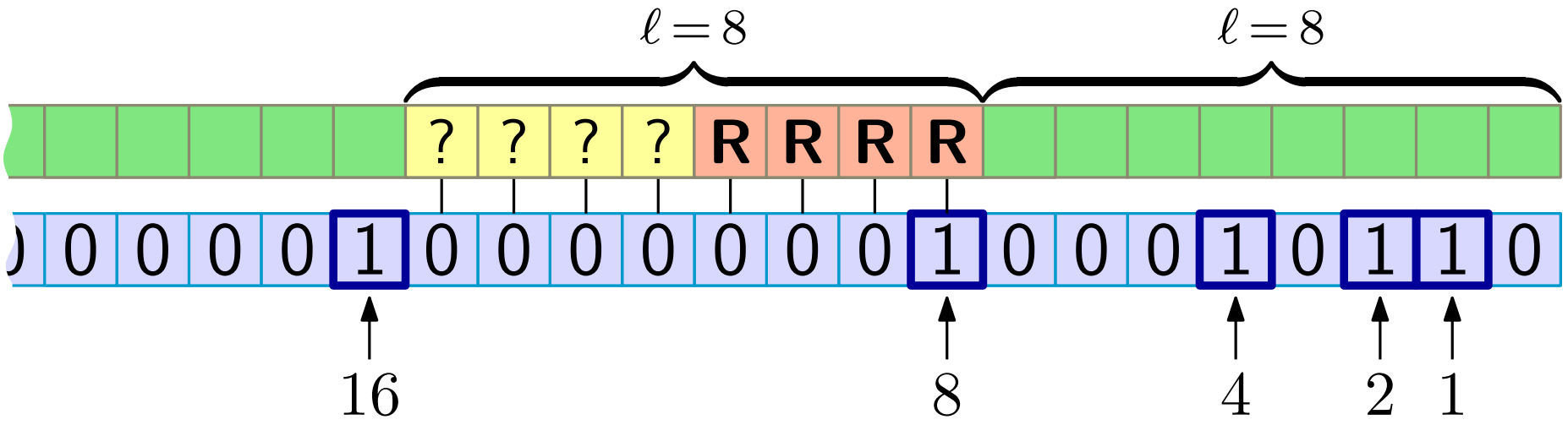

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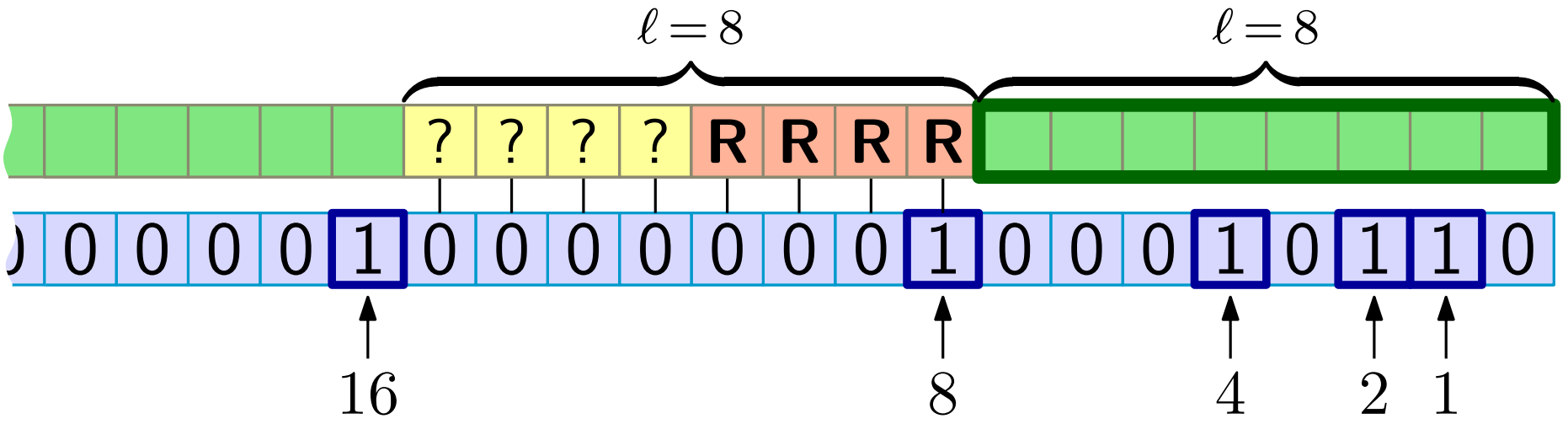
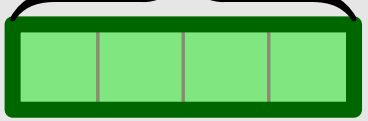
Conclusion: If ℓ is a power of 2 then we recover $\frac{\ell}{2}$ values

Information transfer



The conditional entropy

$H($the outputs during  | all  fixed$) \geqslant \frac{\ell}{2}\delta$

Conclusion: If ℓ is a power of 2 then we recover $\frac{\ell}{2}$ values

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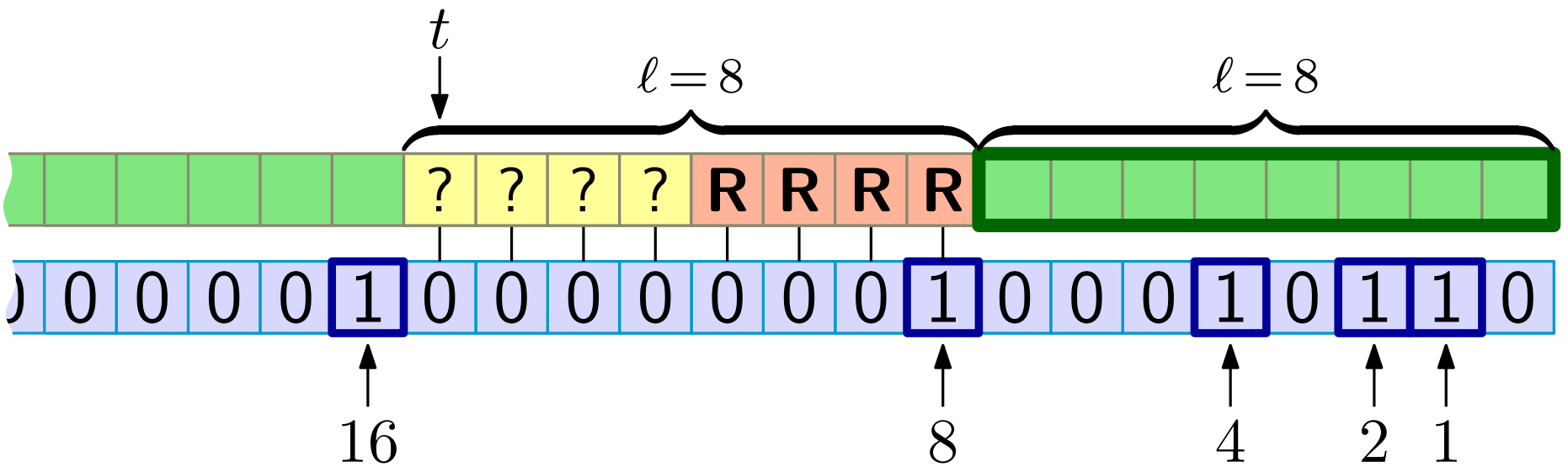


The conditional entropy

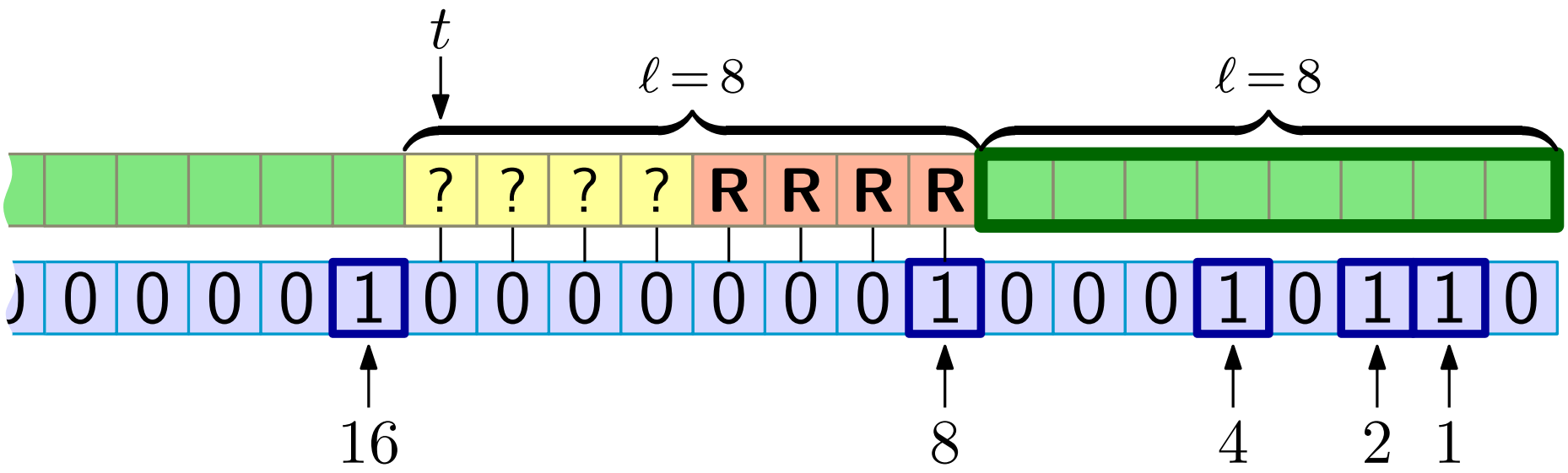
$H($the outputs during $\overbrace{\square\square\square\square}^{\ell}$ | all $\square$ fixed$) \geqslant \frac{\ell}{2}\delta$

The conditional information transfer

$\mathbb{E}\left[|IT(t,\ell)| \mid \text{all } \square \text{ fixed}\right] \geqslant \frac{\delta}{4w}\ell - \frac{1}{2}$

w bits per cell

Information transfer



Suppose that all values (and ?) from the stream are chosen uniformly at random from $[q]$.

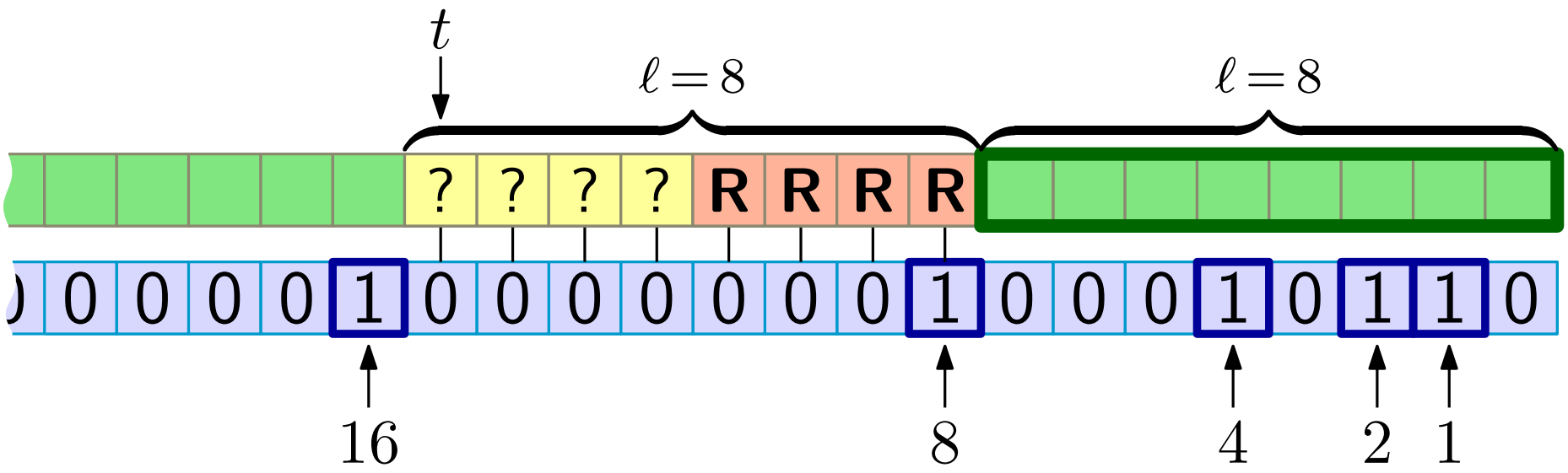
By linearity of expectation...

The conditional information transfer

$$\mathbb{E}\left[|IT(t,\ell)| \mid \text{all fixed}\right] \geqslant \frac{\delta}{4w}\ell - \frac{1}{2}$$

w bits per cell

Information transfer



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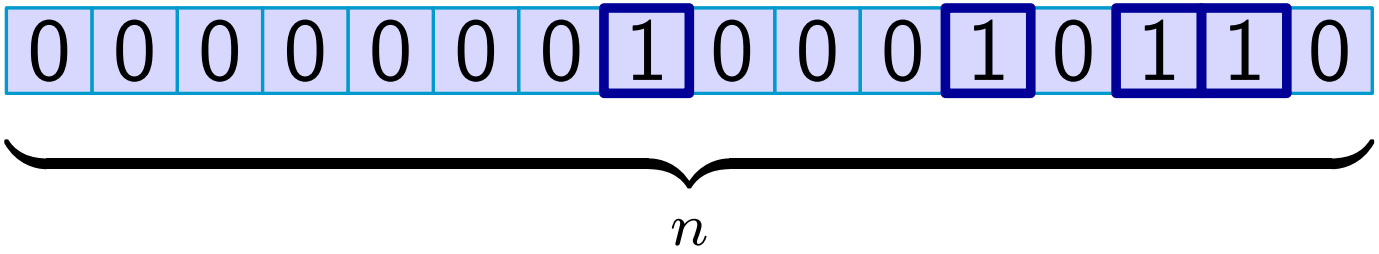
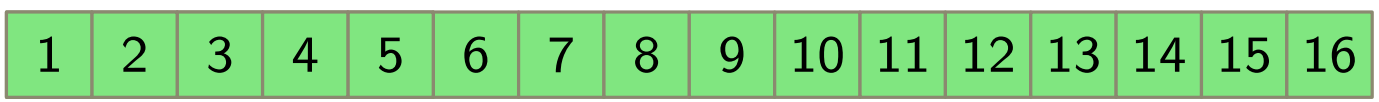
w bits per cell

Total number of cell reads



Feed the algorithm with n values chosen uniformly at random from $[q]$.



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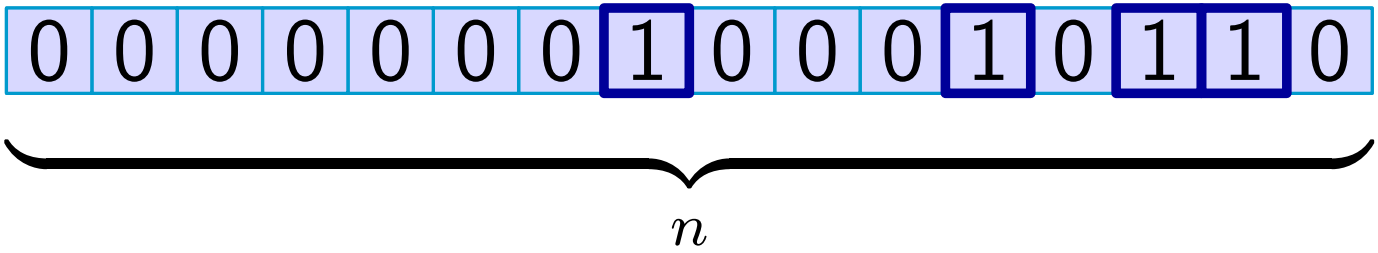

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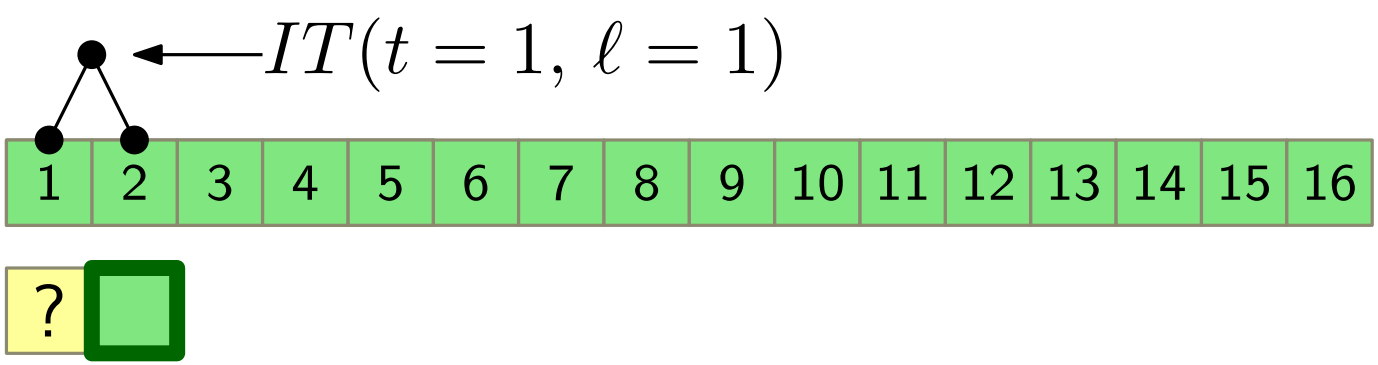

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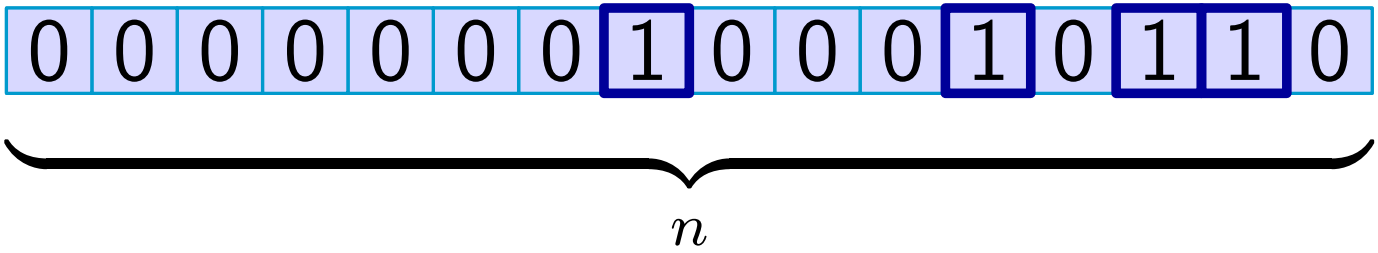

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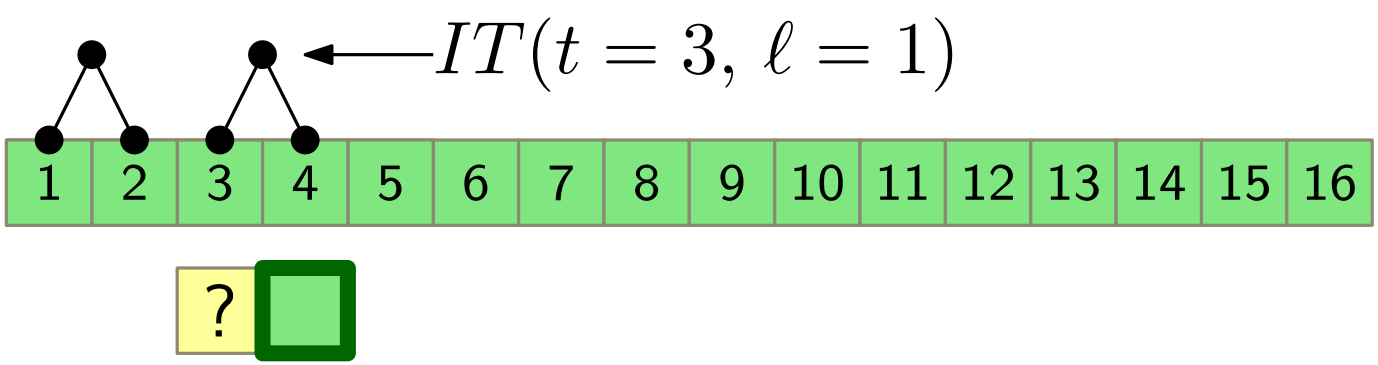

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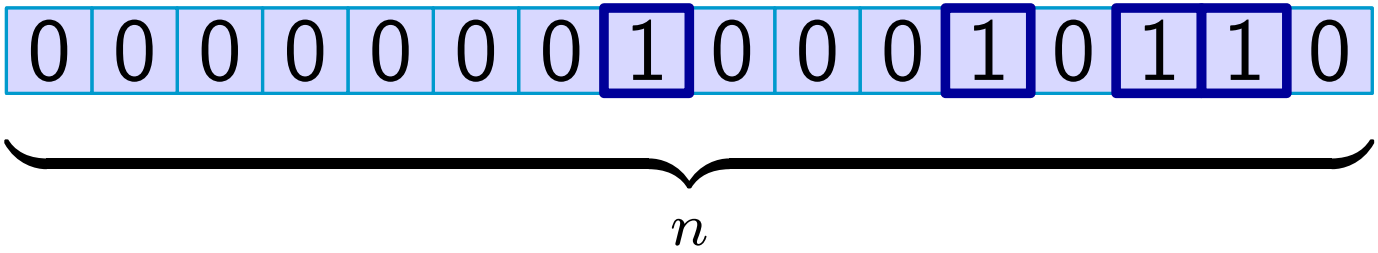

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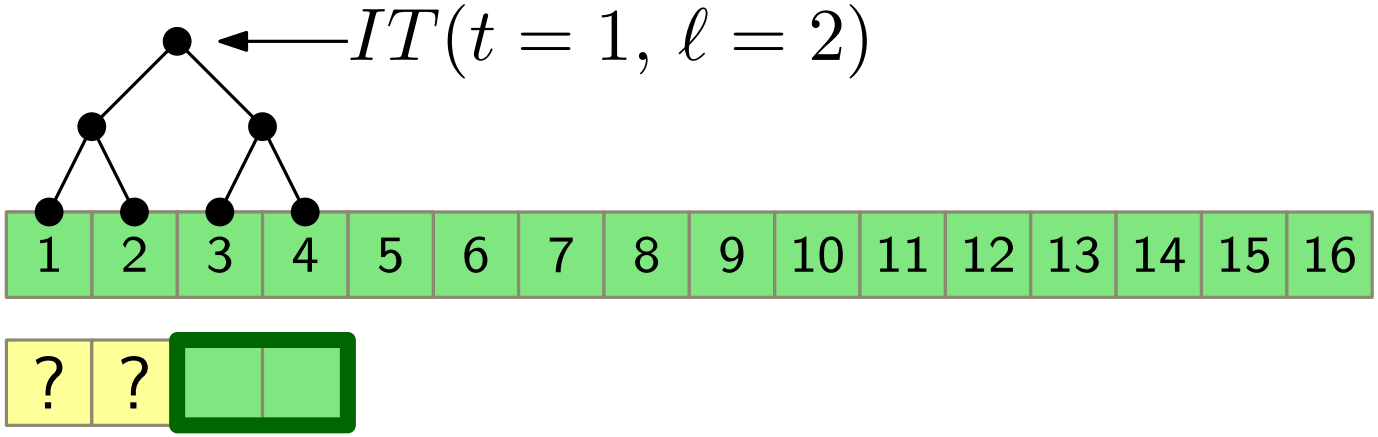

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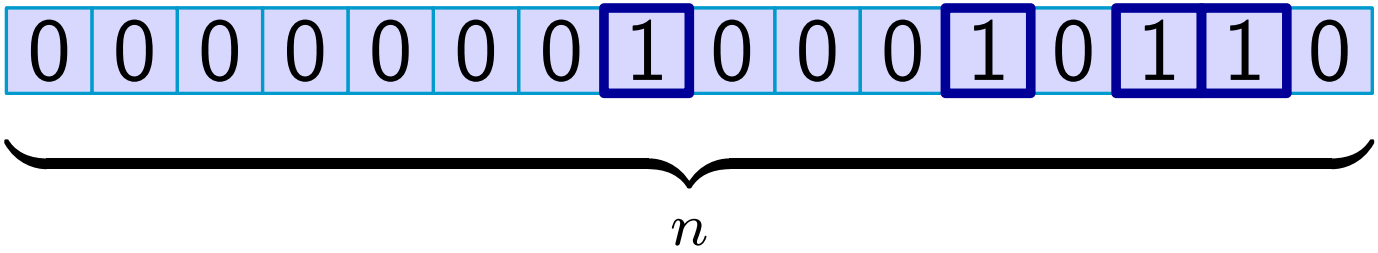

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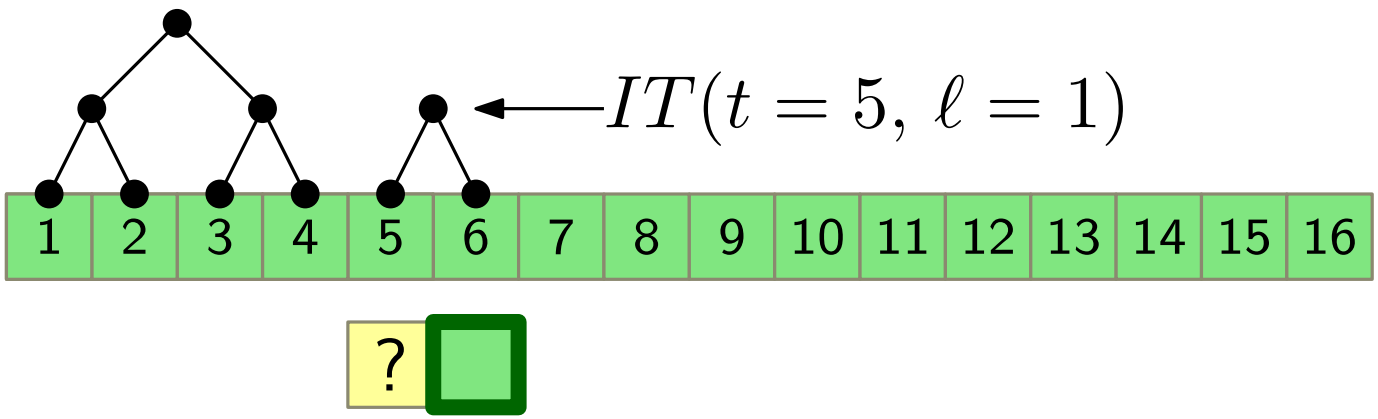

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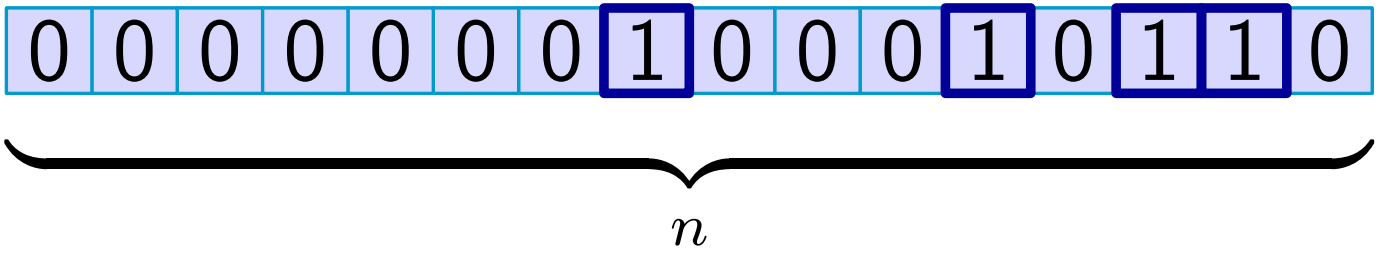

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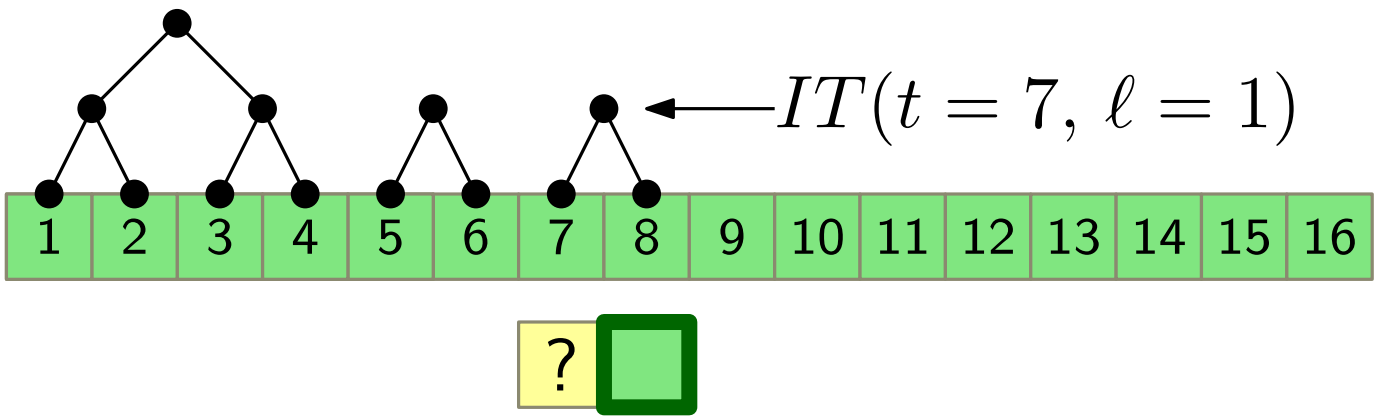

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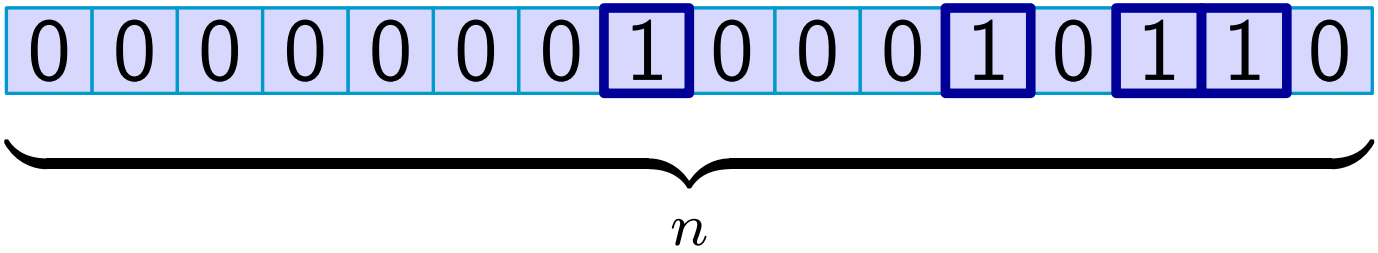
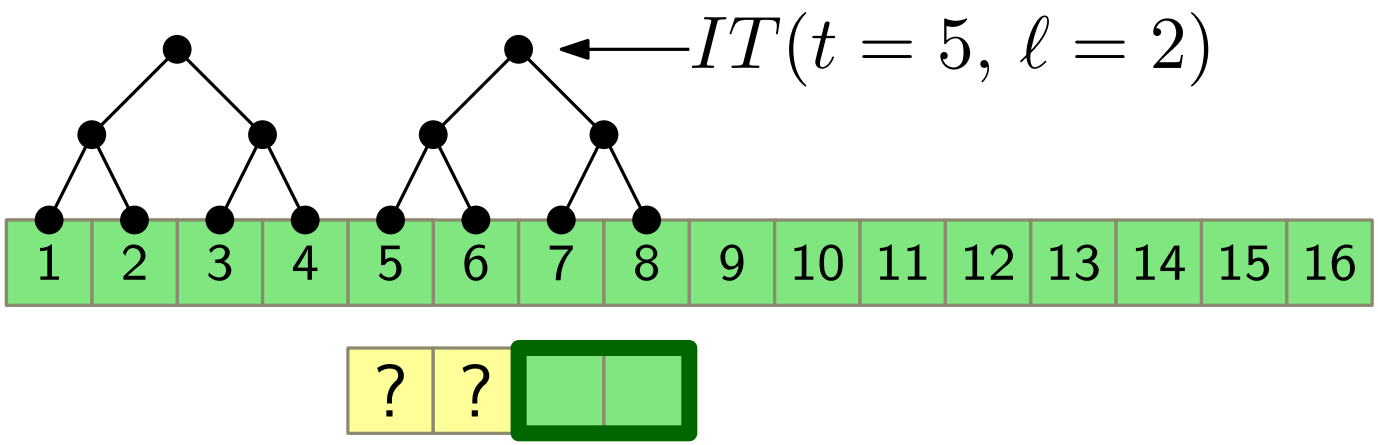
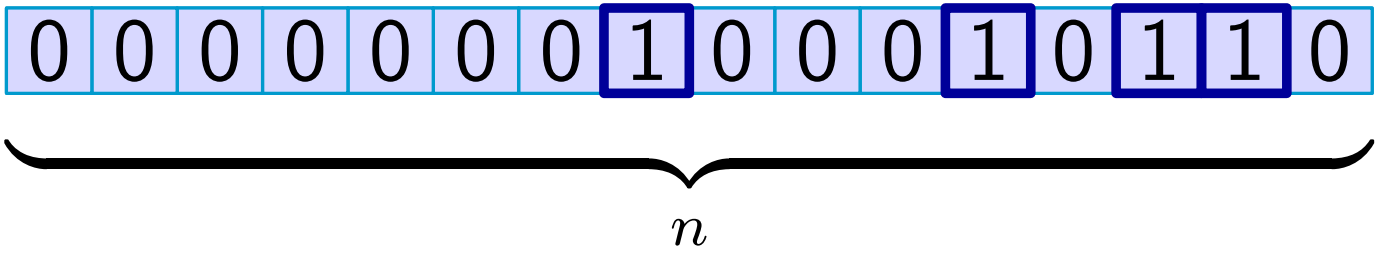


Feed the algorithm with n values chosen uniformly at random from $[q]$.



Total number of cell reads



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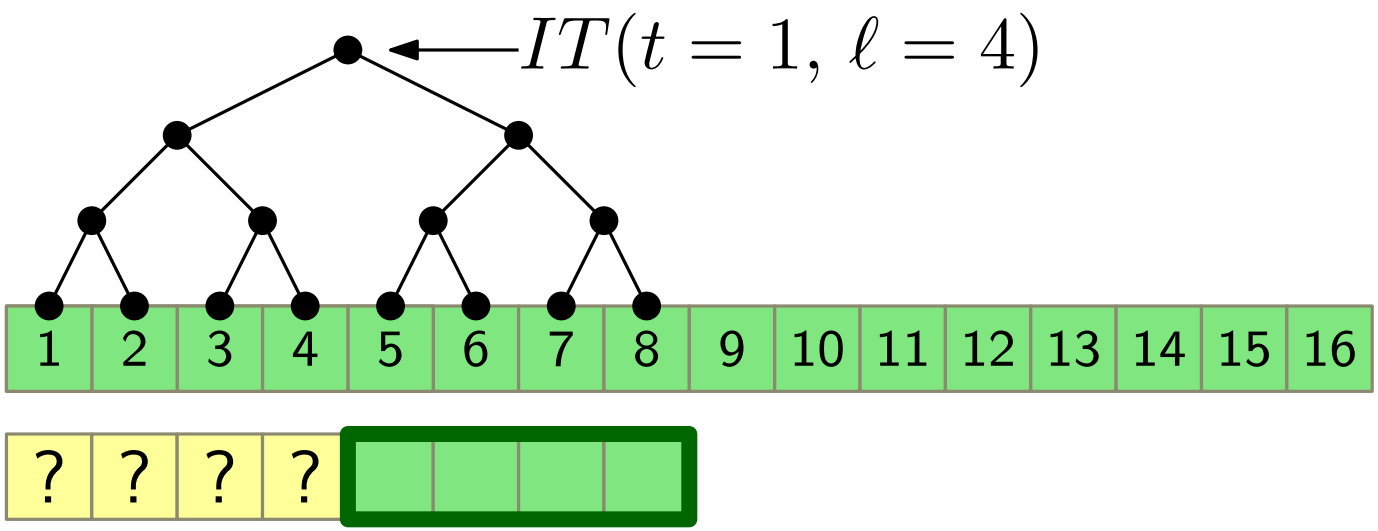
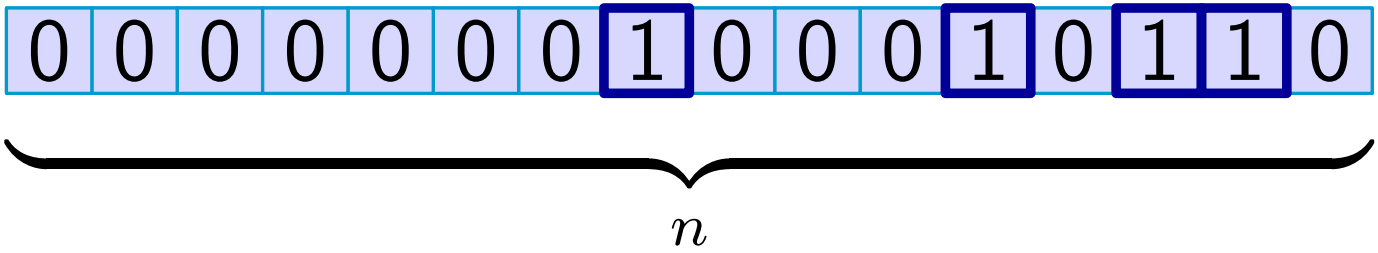
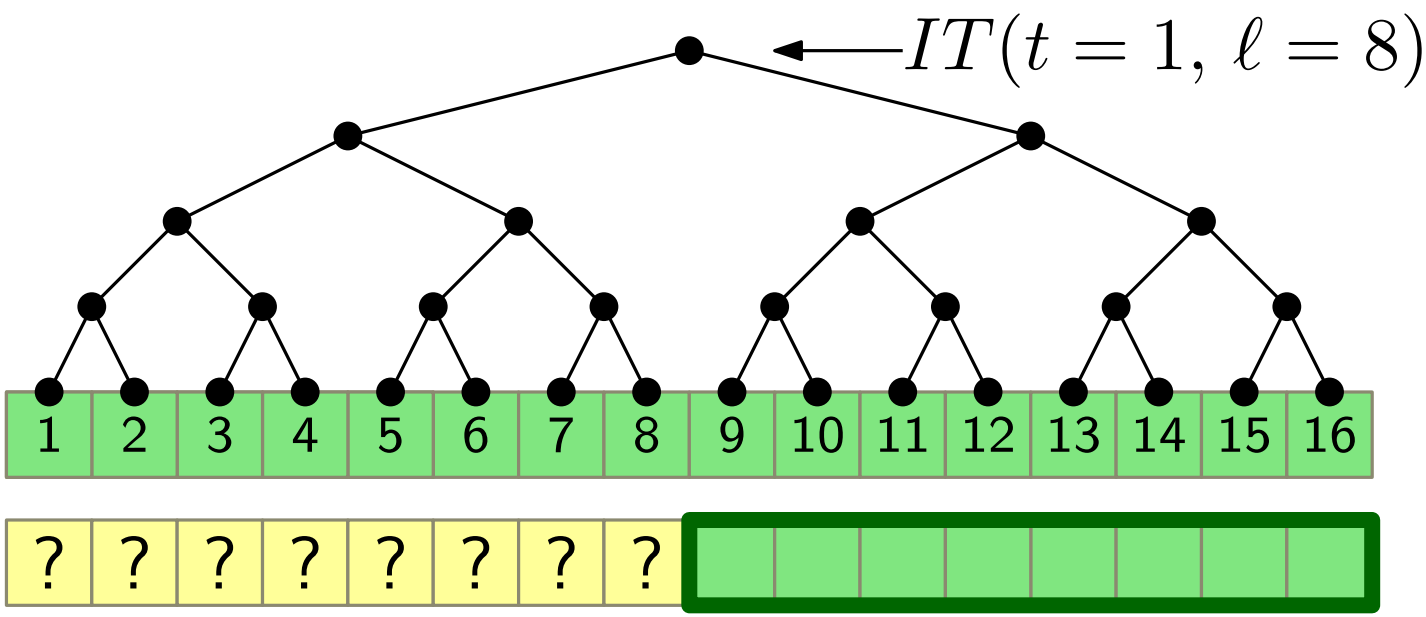


Total number of cell reads



Feed the algorithm with n values chosen uniformly at random from $[q]$.

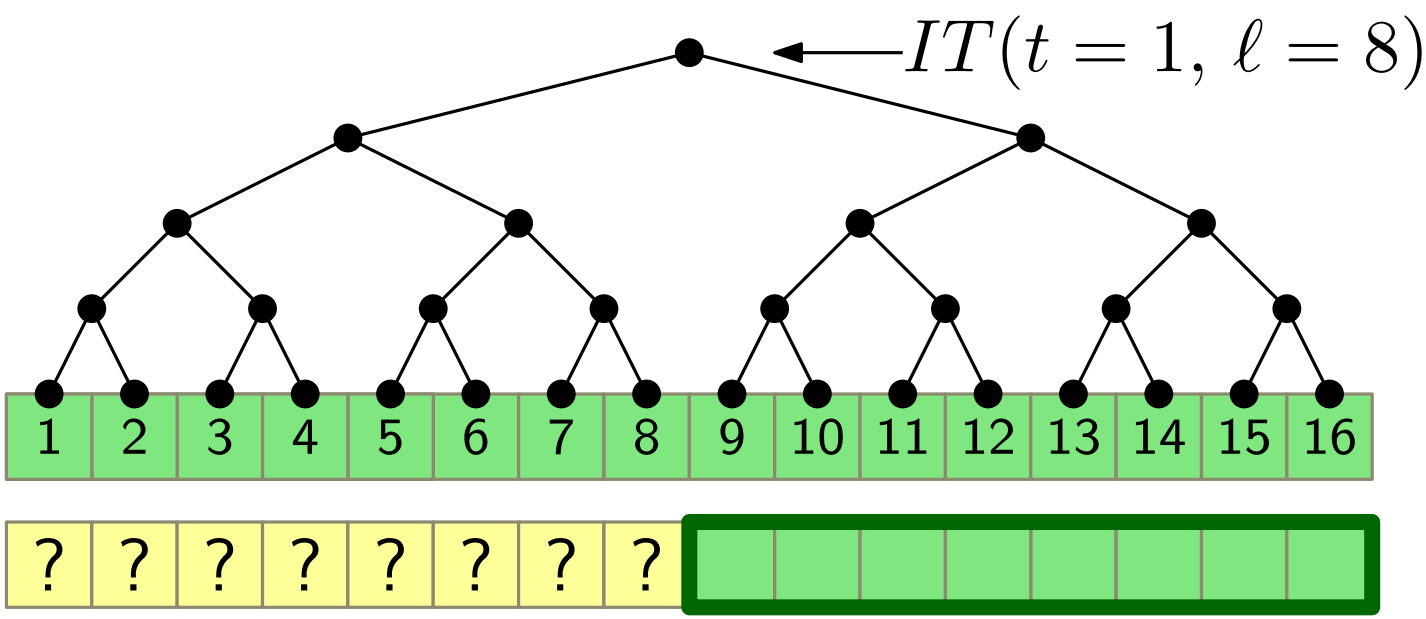


Total number of cell reads

The number of cell reads during the n inputs is at least

$$\sum_{\text{internal node } v} |IT(t_v, \ell_v)|$$

random from $[q]$.



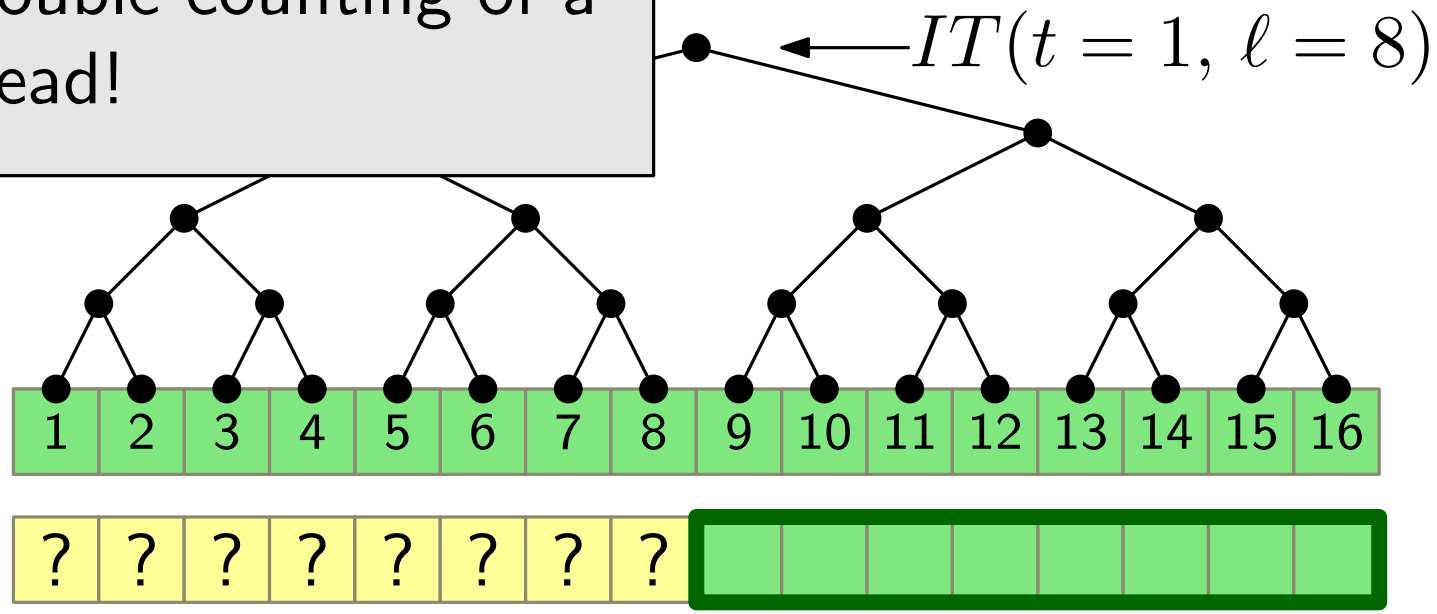
Total number of cell reads

The number of cell reads during the n inputs is at least

$$\sum_{\text{internal node } v} |IT(t_v, \ell_v)|$$

random from $[q]$.

No double counting of a cell read!



Total number of cell reads

The number of cell reads during the n inputs is at least

$$\sum_{\text{internal node } v} |IT(t_v, \ell_v)|$$

The expected number of cell reads is at least

$$\begin{aligned}
\mathbb{E}\left[\sum_{\text{internal node } v} |IT(t_v, \ell_v)|\right] &= \sum_{\text{internal node } v} \mathbb{E}\left[|IT(t_v, \ell_v)|\right] \\
&\geqslant \sum_{\text{internal node } v} \frac{\delta}{4w}\ell_v - \frac{1}{2} \\
&= \Omega\left(\frac{\delta}{w} \cdot n \log n\right)
\end{aligned}$$

Total number of cell reads

The number of cell reads during the n inputs is at least

$$\sum_{\text{internal node } v} |IT(t_v, \ell_v)|$$

The expected number of cell reads is at least

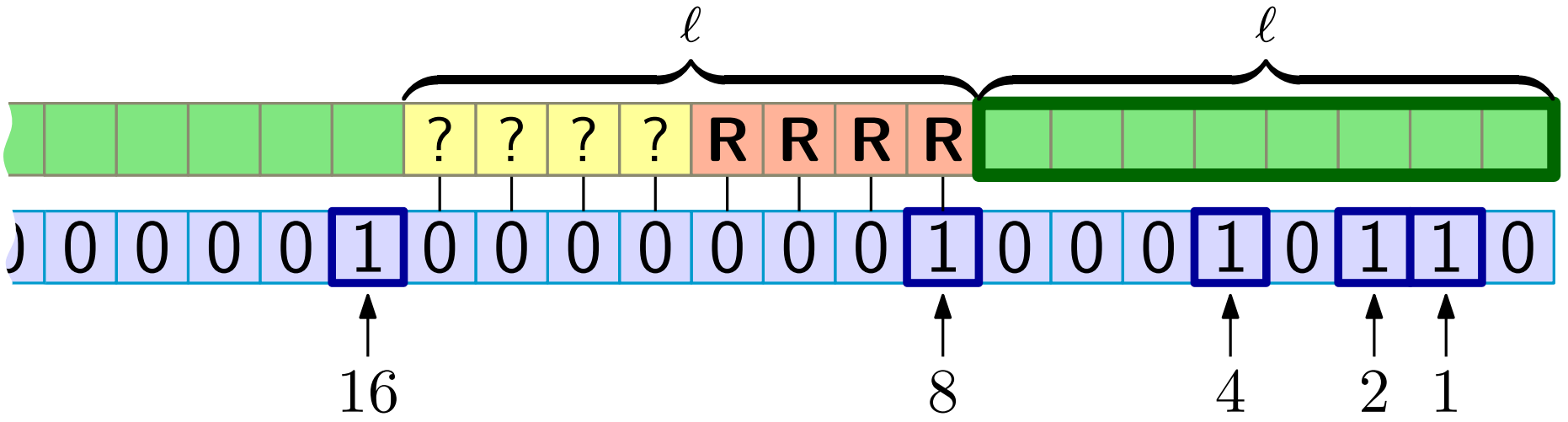
$$\mathbb{E}\left[\sum_{\text{internal node } v} |IT(t_v, \ell_v)|\right] = \sum_{\text{internal node } v} \mathbb{E}\left[|IT(t_v, \ell_v)|\right]$$

$$\geqslant \sum_{\text{internal node } v} \frac{\delta}{4w}\ell_v - \frac{1}{2}$$

$$= \Omega\left(\frac{\delta}{w} \cdot n \log n\right)$$

So...
The amortised time lower bound per output is $\Omega\left(\frac{\delta}{w} \log n\right)$

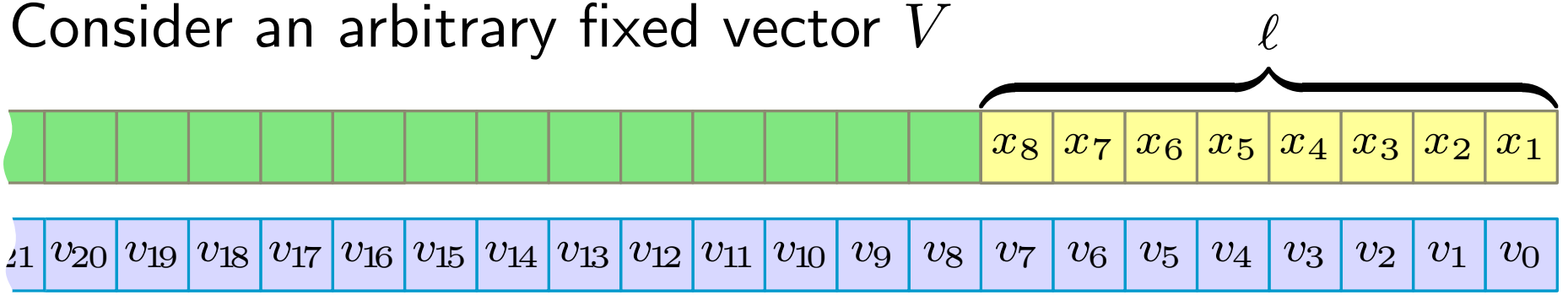
Random fixed vector



If ℓ is a power of 2 then we recover $\frac{\ell}{2}$ values

Random fixed vector

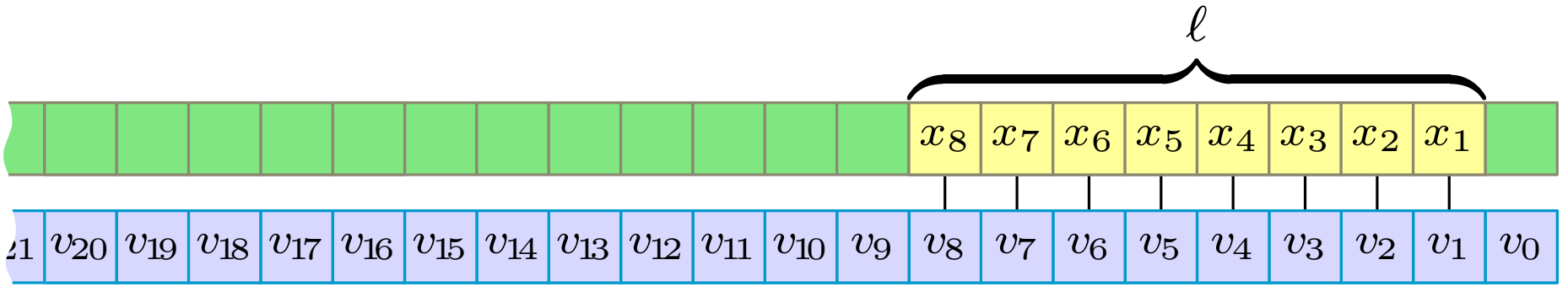
Consider an arbitrary fixed vector V



Fixed value

x_i Unknown value chosen uniformly at random from $[q]$

Random fixed vector



Fixed value

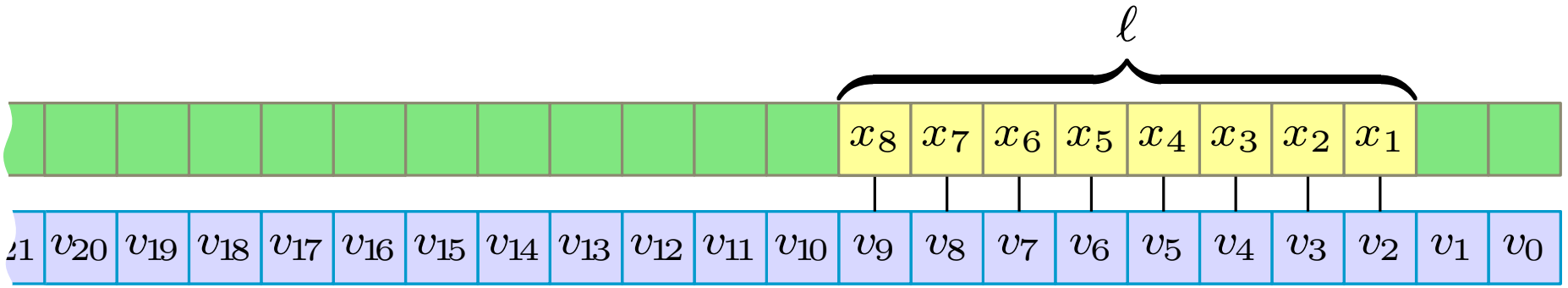
x_i Unknown value chosen uniformly at random from $[q]$

$$\begin{pmatrix} v_8 & v_7 & v_6 & v_5 & v_4 & v_3 & v_2 & v_1 \\ & & & & & & & \\ & & & & & & & \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_8 \\ \\ \end{pmatrix}$$

Contribution from alignment with

x_8	x_7	x_6	x_5	x_4	x_3	x_2	x_1

Random fixed vector



Fixed value

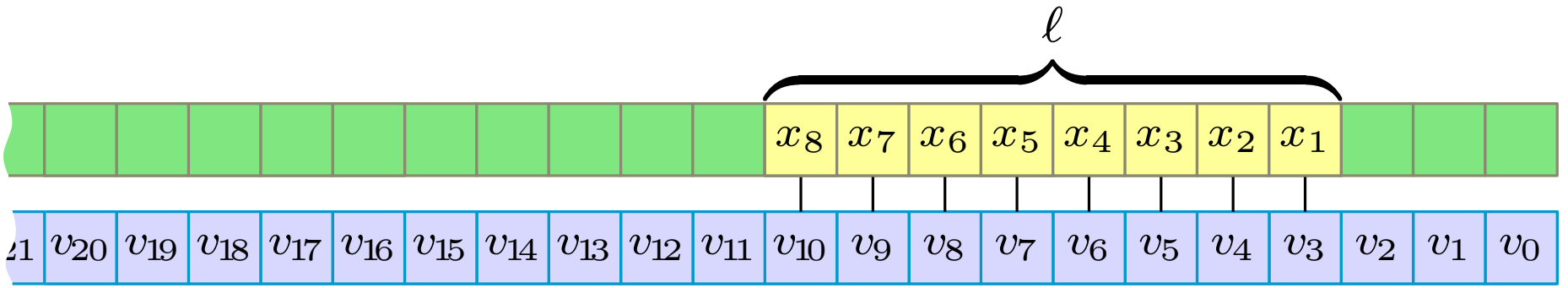
x_i Unknown value chosen uniformly at random from $[q]$

$$\begin{pmatrix} v_9 & v_8 & v_7 & v_6 & v_5 & v_4 & v_3 & v_2 \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_7 \end{pmatrix}$$

Contribution from alignment with

x_8 x_7 x_6 x_5 x_4 x_3 x_2 x_1

Random fixed vector

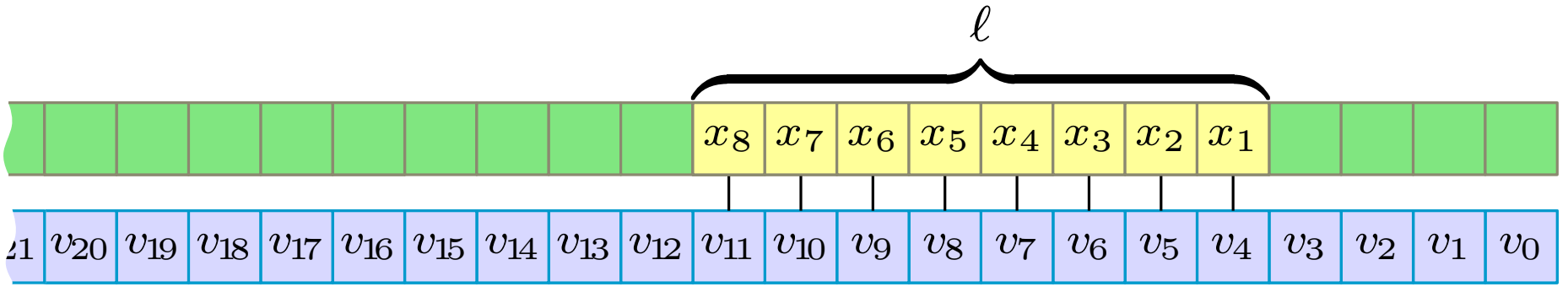


Fixed value

x_i Unknown value chosen uniformly at random from $[q]$

$$\begin{pmatrix} v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 & v_4 & v_3 \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \\ y_6 \\ \\ \end{pmatrix}$$

Random fixed vector

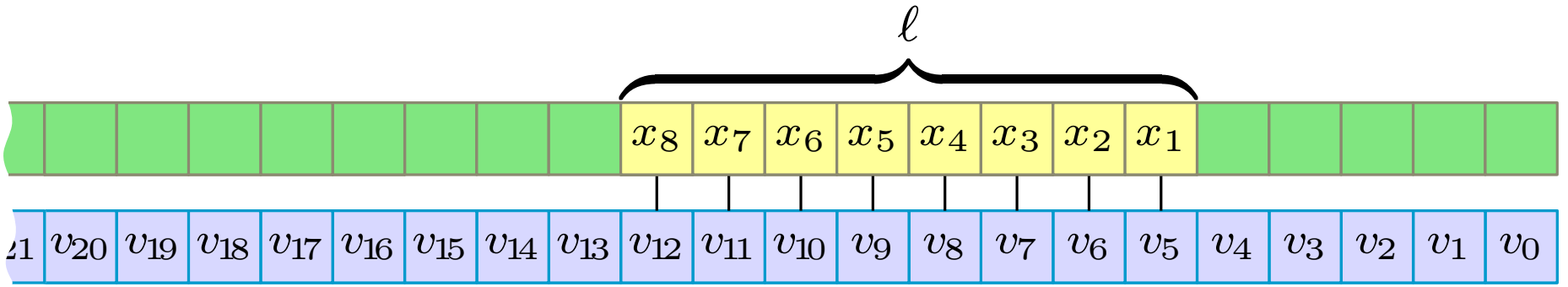


Fixed value

x_i Unknown value chosen uniformly at random from $[q]$

$$\begin{pmatrix} v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 & v_4 \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \\ \\ \\ \\ \end{pmatrix} y_5$$

Random fixed vector

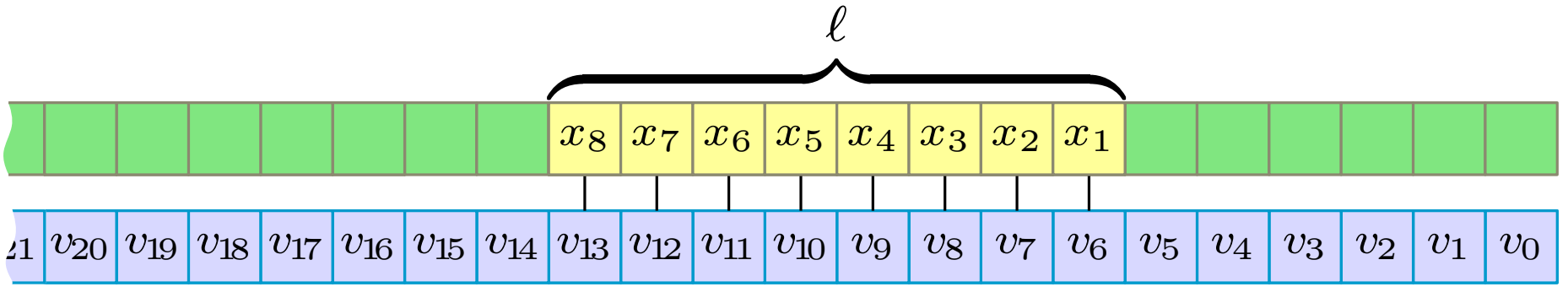


Fixed value

x_i Unknown value chosen uniformly at random from $[q]$

$$\begin{pmatrix} v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \\ y_4 \\ \\ \end{pmatrix}$$

Random fixed vector

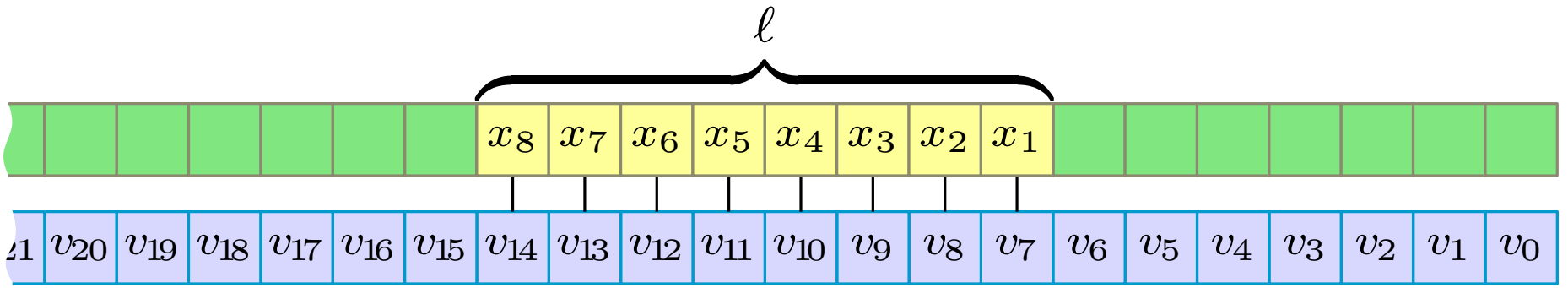


Fixed value

x_i Unknown value chosen uniformly at random from $[q]$

$$\begin{pmatrix} & & & & & & & \\ v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 \\ & & & & & & & \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \\ y_3 \\ \\ \end{pmatrix}$$

Random fixed vector

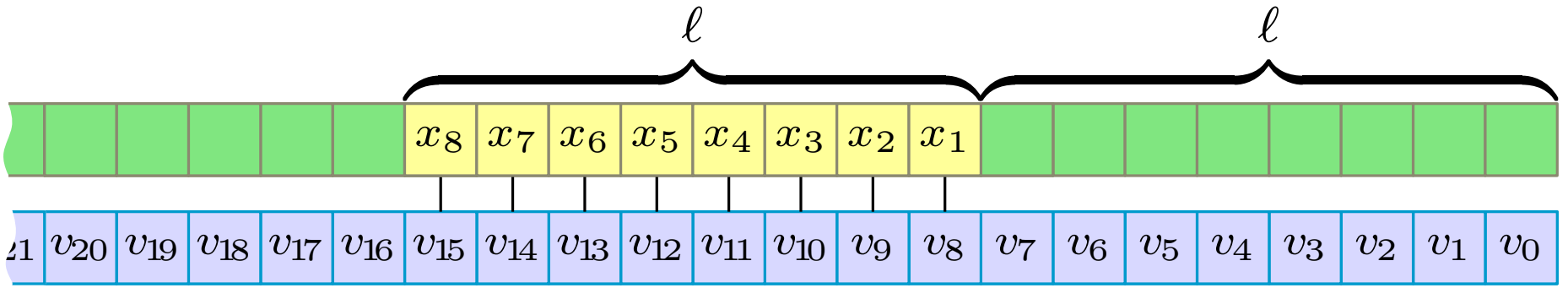


Fixed value

x_i Unknown value chosen uniformly at random from $[q]$

$$\begin{pmatrix} & & & & & & & \\ v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 \\ & & & & & & & \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \\ \\ \\ \\ y_2 \\ \\ \end{pmatrix}$$

Random fixed vector

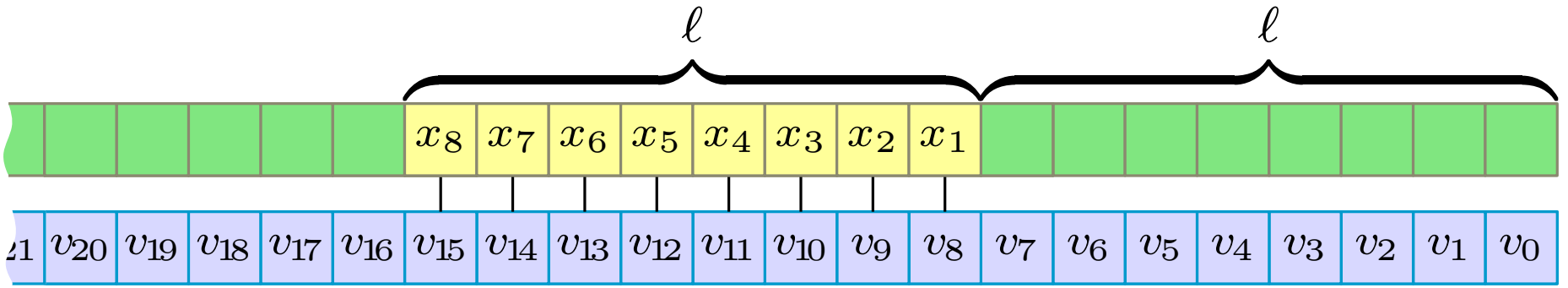


Fixed value

x_i Unknown value chosen uniformly at random from $[q]$

$$\begin{pmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ v_{15} & v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \\ \\ \\ \\ y_1 \end{pmatrix}$$

Random fixed vector



Fixed value

x_i Unknown value chosen uniformly at random from $[q]$

$$\begin{pmatrix} v_8 & v_7 & v_6 & v_5 & v_4 & v_3 & v_2 & v_1 \\ v_9 & v_8 & v_7 & v_6 & v_5 & v_4 & v_3 & v_2 \\ v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 & v_4 & v_3 \\ v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 & v_4 \\ v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 \\ v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 \\ v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 \\ v_{15} & v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_8 \\ y_7 \\ y_6 \\ y_5 \\ y_4 \\ y_3 \\ y_2 \\ y_1 \end{pmatrix}$$

Random fixed vector

The matrix is a Toeplitz matrix.
Suppose we choose the vector V randomly from $[q]^n$.

$$\begin{pmatrix} v_8 & v_7 & v_6 & v_5 & v_4 & v_3 & v_2 & v_1 \\ v_9 & v_8 & v_7 & v_6 & v_5 & v_4 & v_3 & v_2 \\ v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 & v_4 & v_3 \\ v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 & v_4 \\ v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 \\ v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 \\ v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 \\ v_{15} & v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_8 \\ y_7 \\ y_6 \\ y_5 \\ y_4 \\ y_3 \\ y_2 \\ y_1 \end{pmatrix}$$

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If q is a prime then we operate in the field $\mathbb{Z}/q\mathbb{Z}$, and the Toeplitz matrix is invertible with probability $1 - 1/q \geqslant 1/2$.

$$\begin{pmatrix} v_8 & v_7 & v_6 & v_5 & v_4 & v_3 & v_2 & v_1 \\ v_9 & v_8 & v_7 & v_6 & v_5 & v_4 & v_3 & v_2 \\ v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 & v_4 & v_3 \\ v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 & v_4 \\ v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 \\ v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 \\ v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 \\ v_{15} & v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_8 \\ y_7 \\ y_6 \\ y_5 \\ y_4 \\ y_3 \\ y_2 \\ y_1 \end{pmatrix}$$

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When invertible, we can recover $x_1, \ldots, x_\ell$ (ℓ values).

$$\begin{pmatrix} v_8 & v_7 & v_6 & v_5 & v_4 & v_3 & v_2 & v_1 \\ v_9 & v_8 & v_7 & v_6 & v_5 & v_4 & v_3 & v_2 \\ v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 & v_4 & v_3 \\ v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 & v_4 \\ v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 \\ v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 \\ v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 \\ v_{15} & v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 \end{pmatrix} \times \begin{pmatrix} x_8 \\ x_7 \\ x_6 \\ x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_8 \\ y_7 \\ y_6 \\ y_5 \\ y_4 \\ y_3 \\ y_2 \\ y_1 \end{pmatrix}$$

Random fixed vector

The matrix is a Toeplitz matrix.
Suppose we choose the vector V randomly from $[q]^n$.

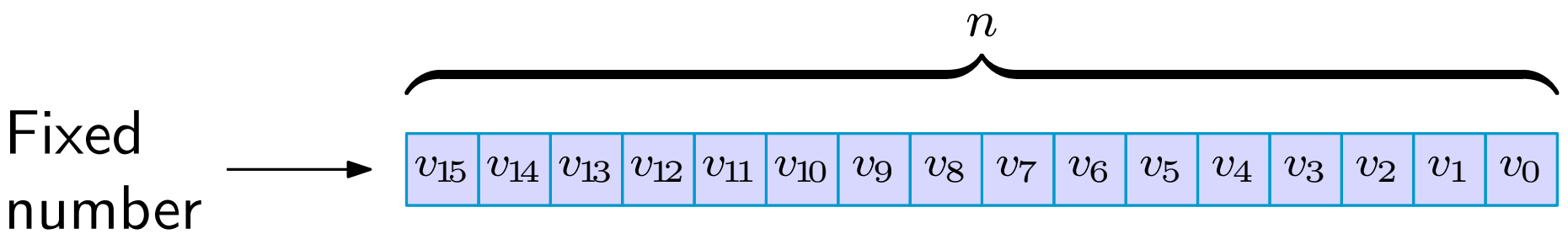
If q is a prime then we operate in the field $\mathbb{Z}/q\mathbb{Z}$, and the Toeplitz matrix is invertible with probability $1 - 1/q \geqslant 1/2$.

When invertible, we can recover $x_1, \ldots, x_\ell$ (ℓ values).

So...
For a random fixed vector V, the amortised time lower bound per output is also $\Omega\left(\frac{\delta}{w} \log n\right)$

$$\begin{pmatrix} v_8 & & & & & & & \\ v_9 & & & & & & & \\ v_{10} & & & & & & & \\ v_{11} & & & & & & & \\ v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 & v_5 \\ v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 & v_6 \\ v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 & v_7 \\ v_{15} & v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 \end{pmatrix} \times \begin{pmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_8 \\ y_7 \\ y_6 \\ y_5 \\ y_4 \\ y_3 \\ y_2 \\ y_1 \end{pmatrix}$$

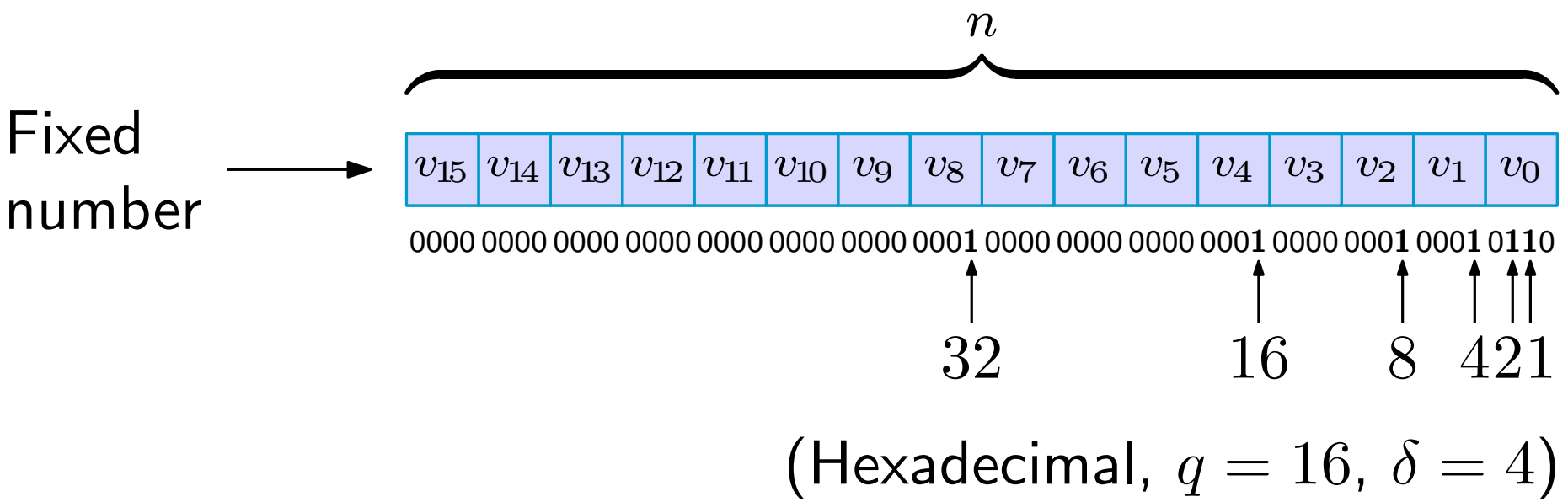
Multiplication



Digits from the set $[q]$ (base q)

$\delta = \log q$ bits per digit

Multiplication

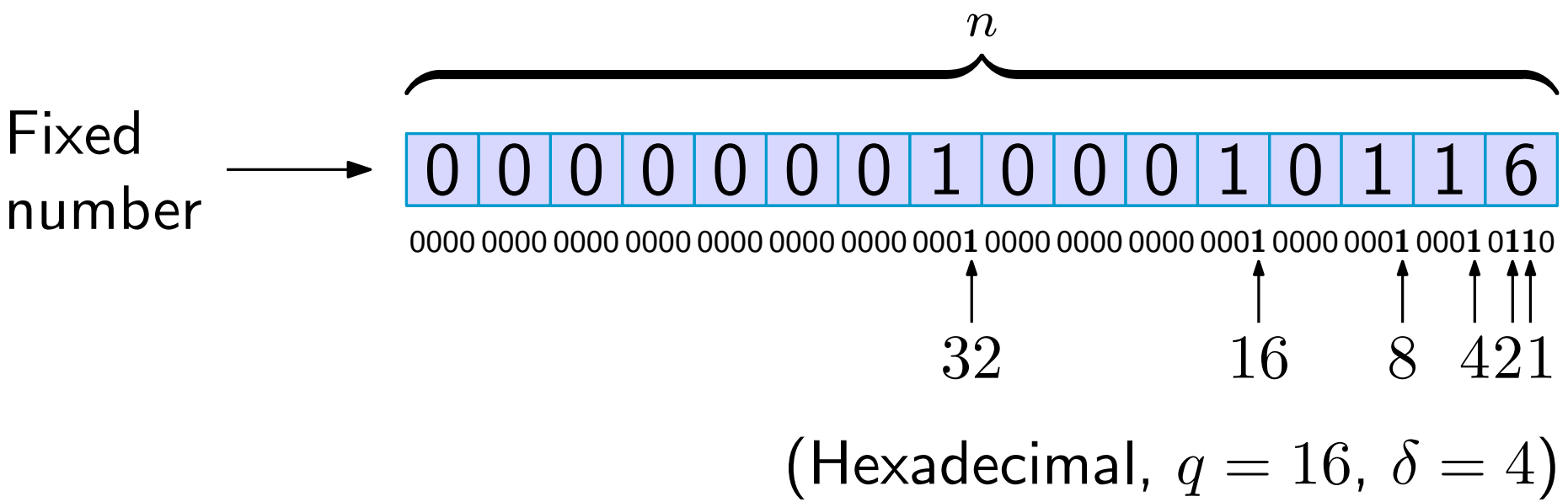


(Hexadecimal, $q = 16$, $\delta = 4$)

Digits from the set $[q]$ (base q)

$\delta = \log q$ bits per digit

Multiplication

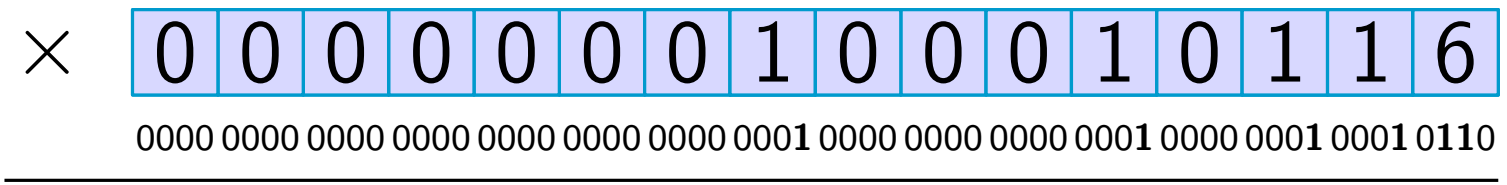


(Hexadecimal, $q = 16$, $\delta = 4$)

Digits from the set $[q]$ (base q)

$\delta = \log q$ bits per digit

Multiplication

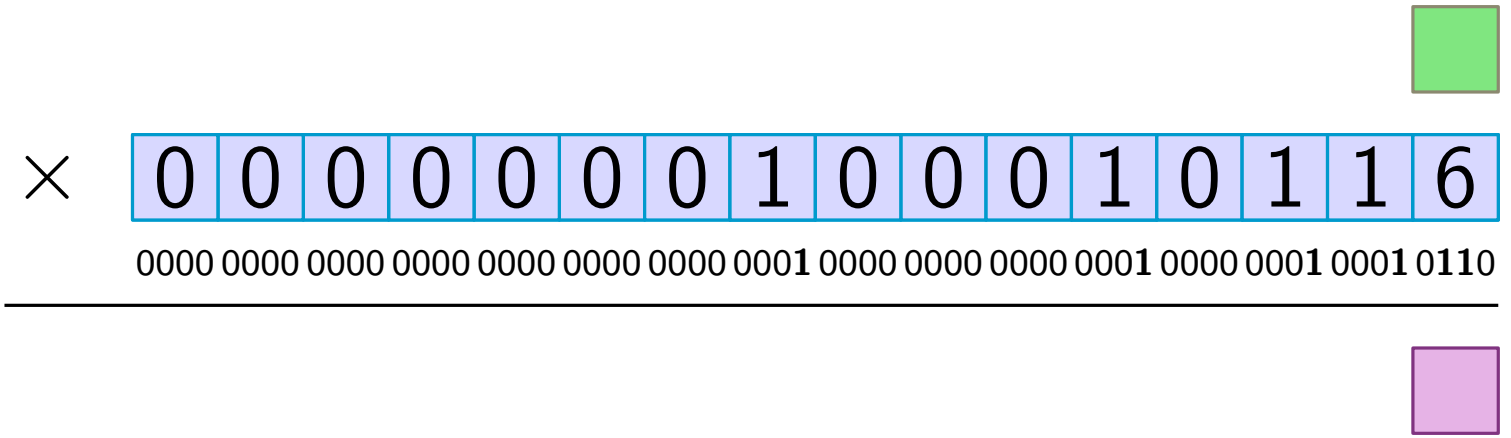


Arriving digit (fixed value)

? Unknown digit chosen uniformly at random from $[q]$

Output digit

Multiplication



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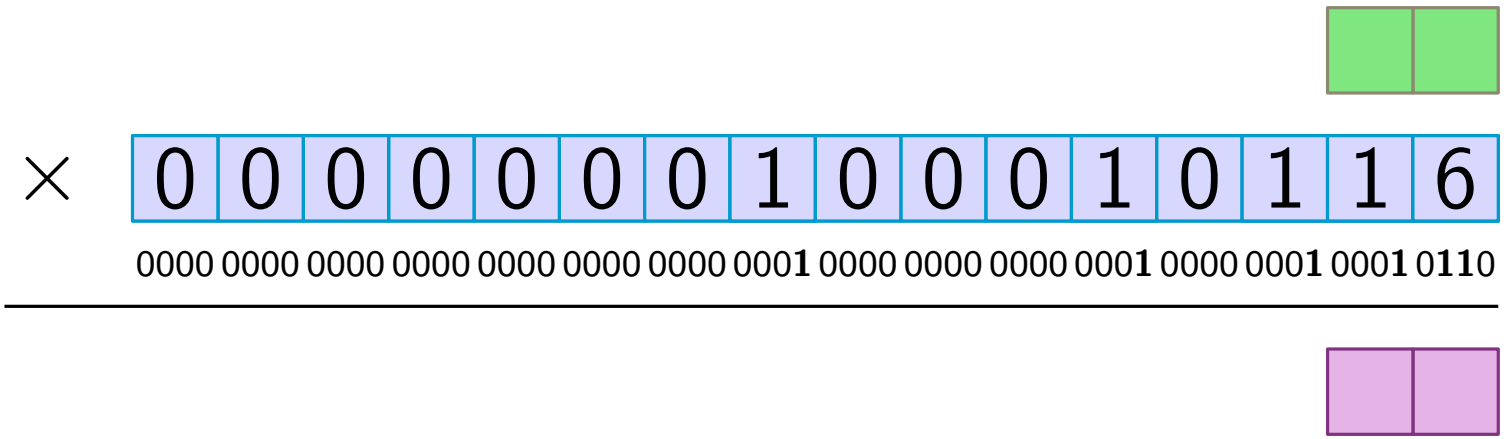
Multiplication



Arriving digit (fixed value)

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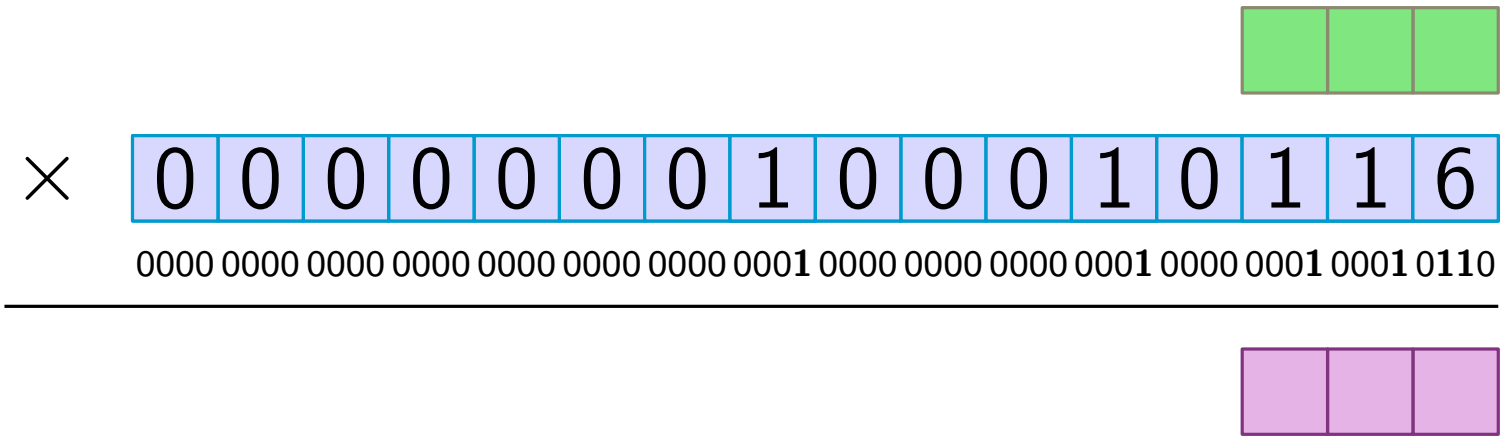
Multiplication

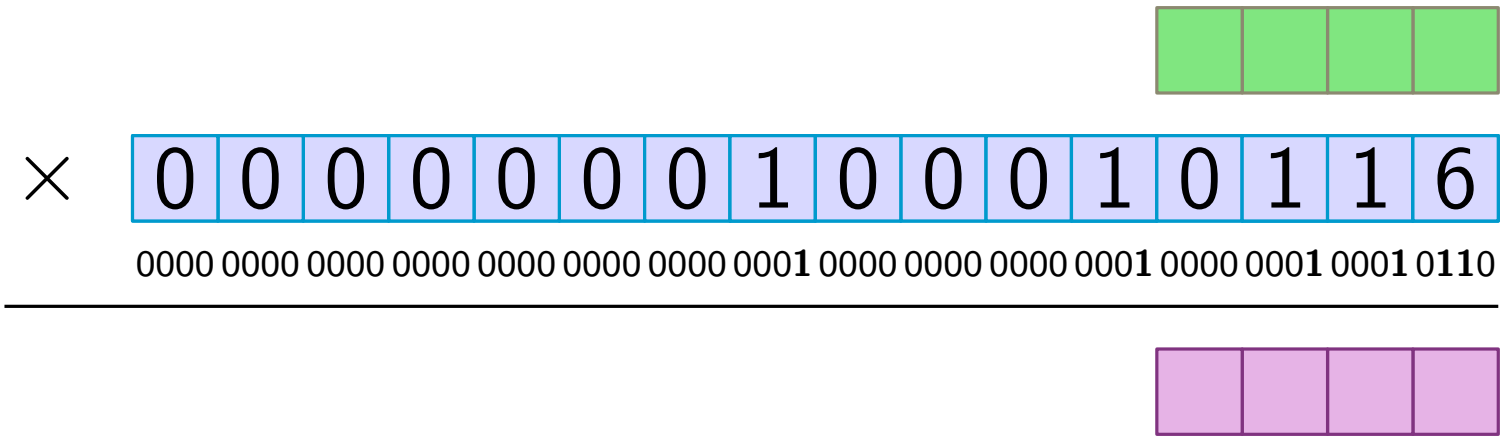


Arriving digit (fixed value)

? Unknown digit chosen uniformly at random from $[q]$

Output digit

Multiplication

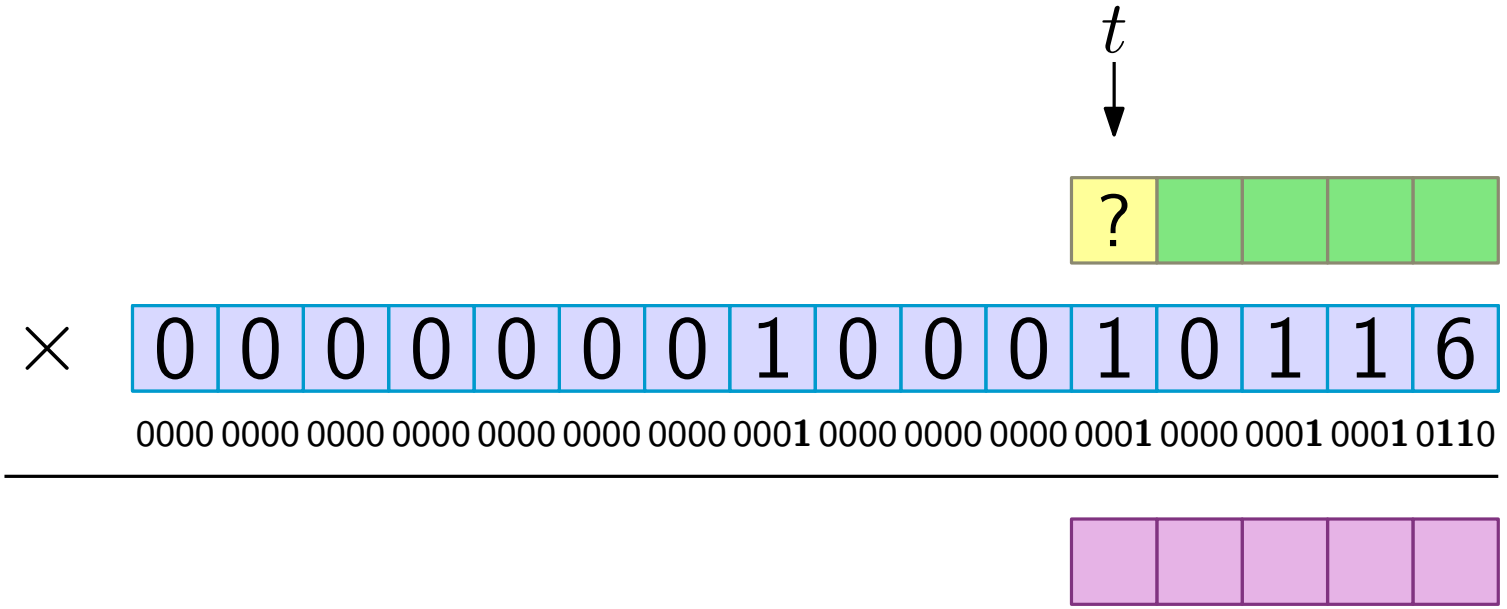


Arriving digit (fixed value)

? Unknown digit chosen uniformly at random from $[q]$

Output digit

Multiplication



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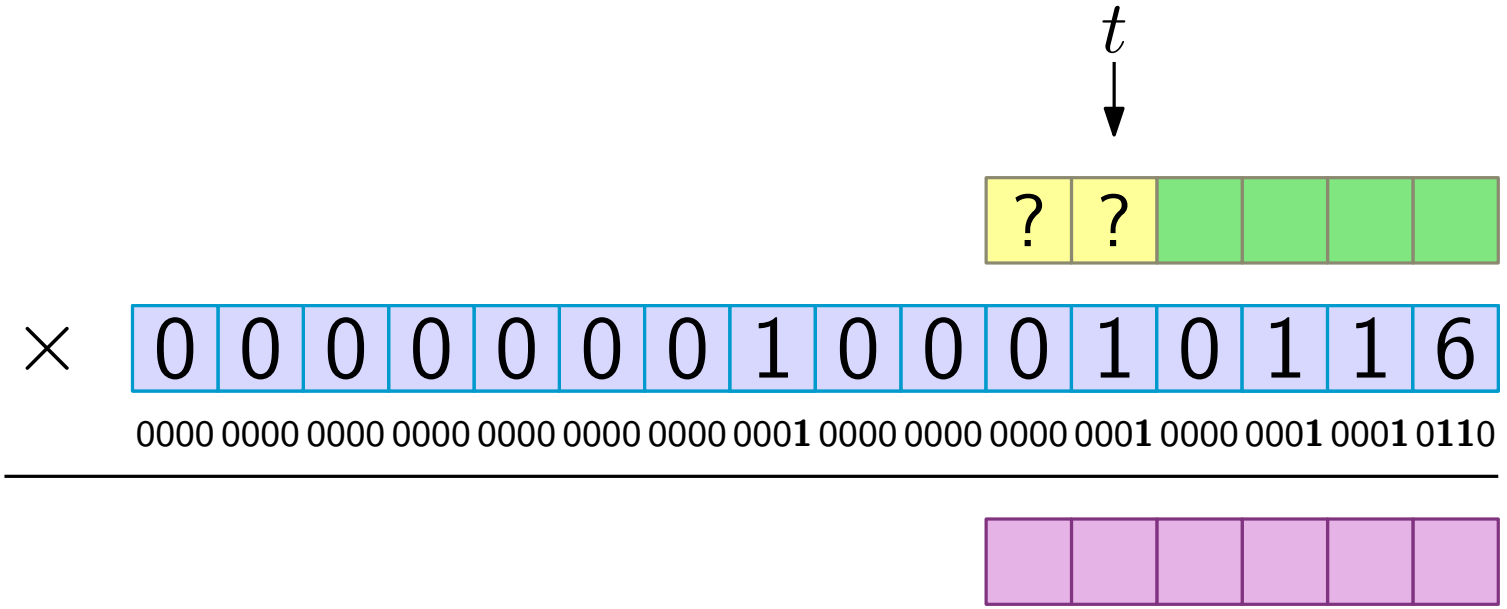
? Unknown digit chosen uniformly at random from $[q]$

Output digit

Multiplication

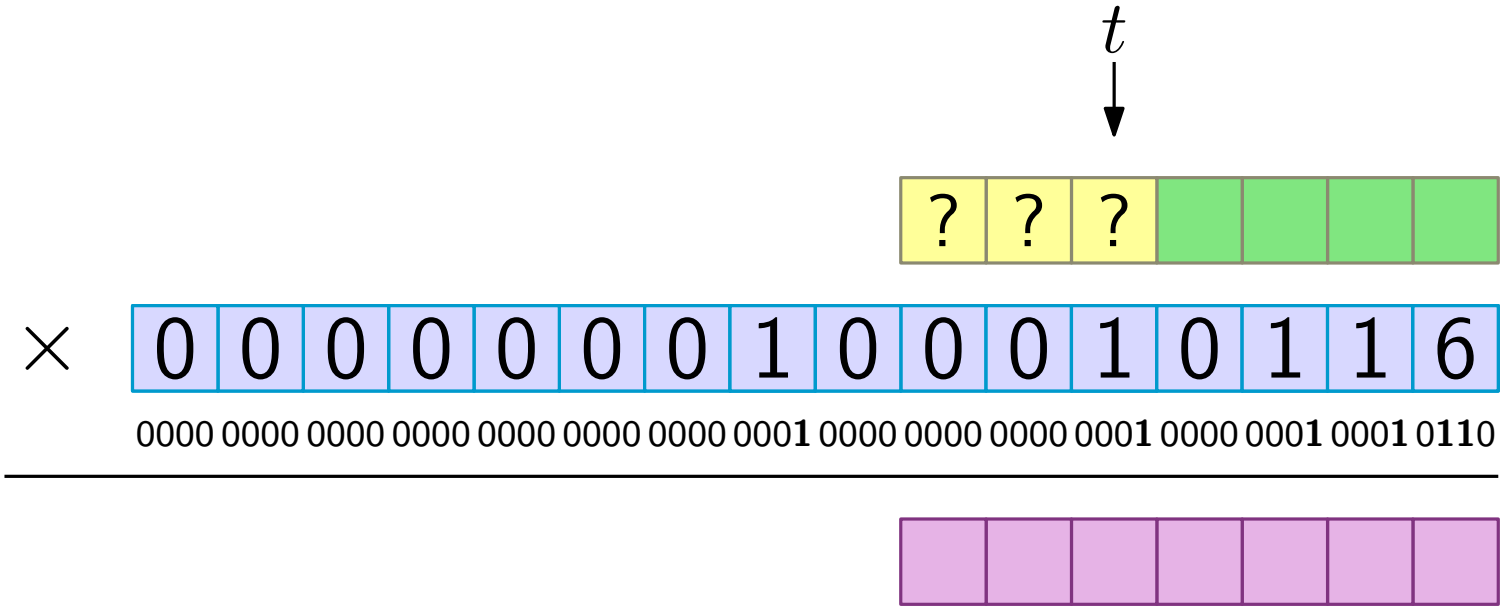


Arriving digit (fixed value)

? Unknown digit chosen uniformly at random from $[q]$

Output digit

Multiplication

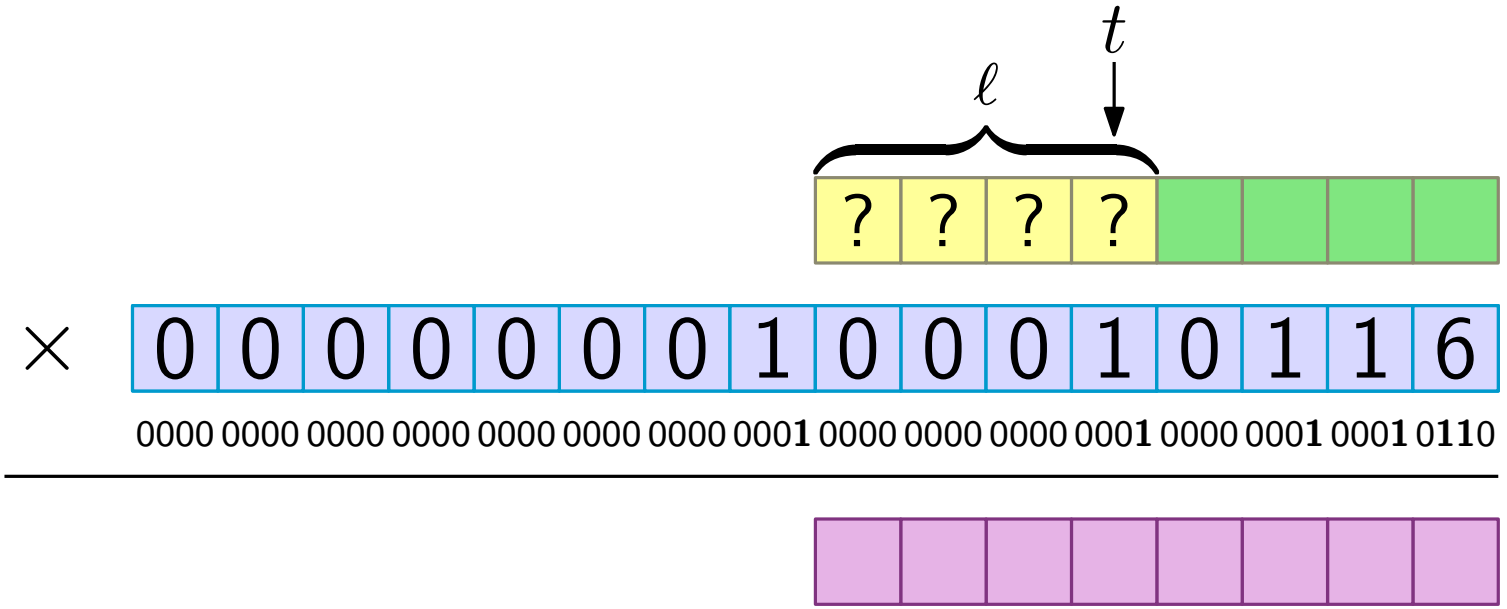


Arriving digit (fixed value)

? Unknown digit chosen uniformly at random from $[q]$

Output digit

Multiplication

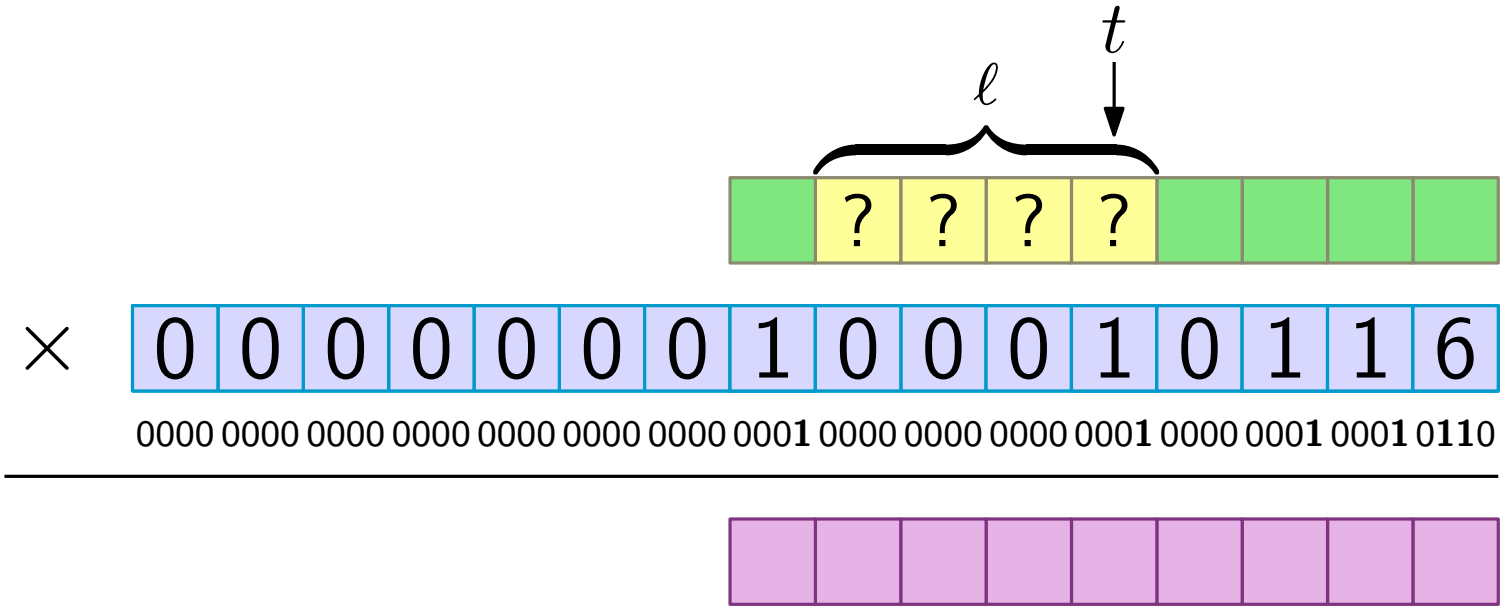


Arriving digit (fixed value)

? Unknown digit chosen uniformly at random from $[q]$

Output digit

Multiplication

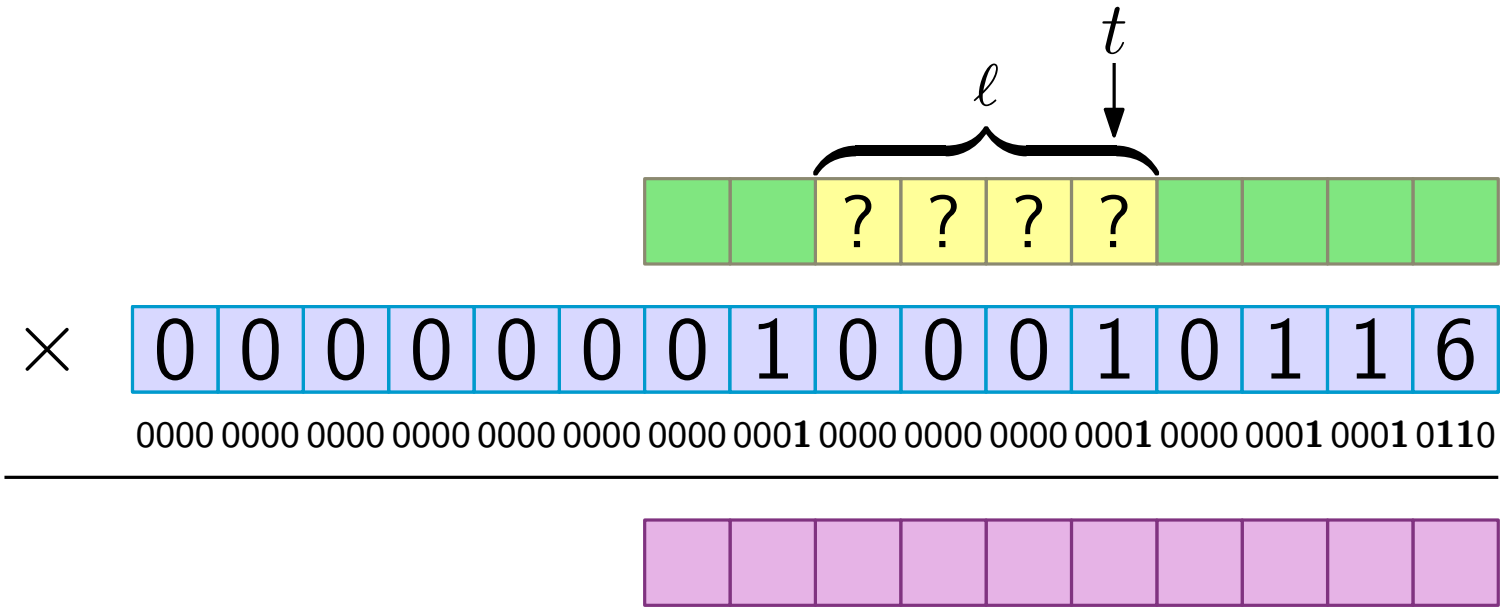


Arriving digit (fixed value)

? Unknown digit chosen uniformly at random from $[q]$

Output digit

Multiplication

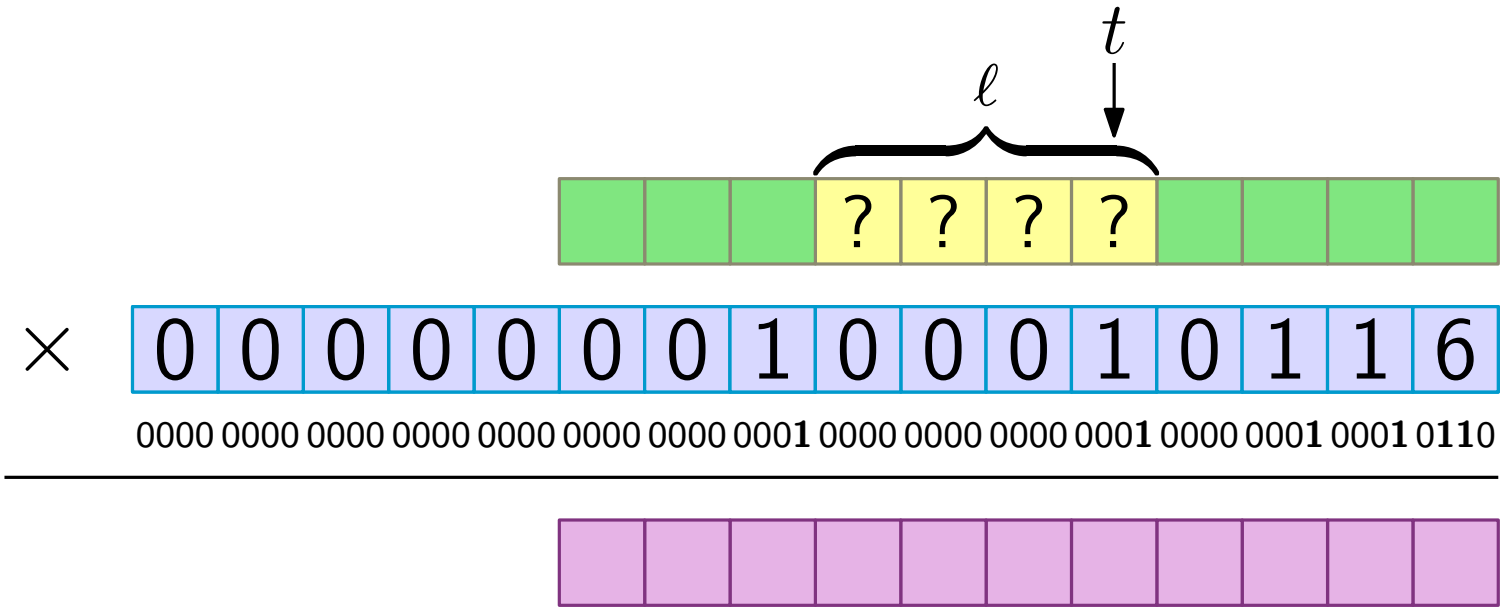


Arriving digit (fixed value)

? Unknown digit chosen uniformly at random from $[q]$

Output digit

Multiplication

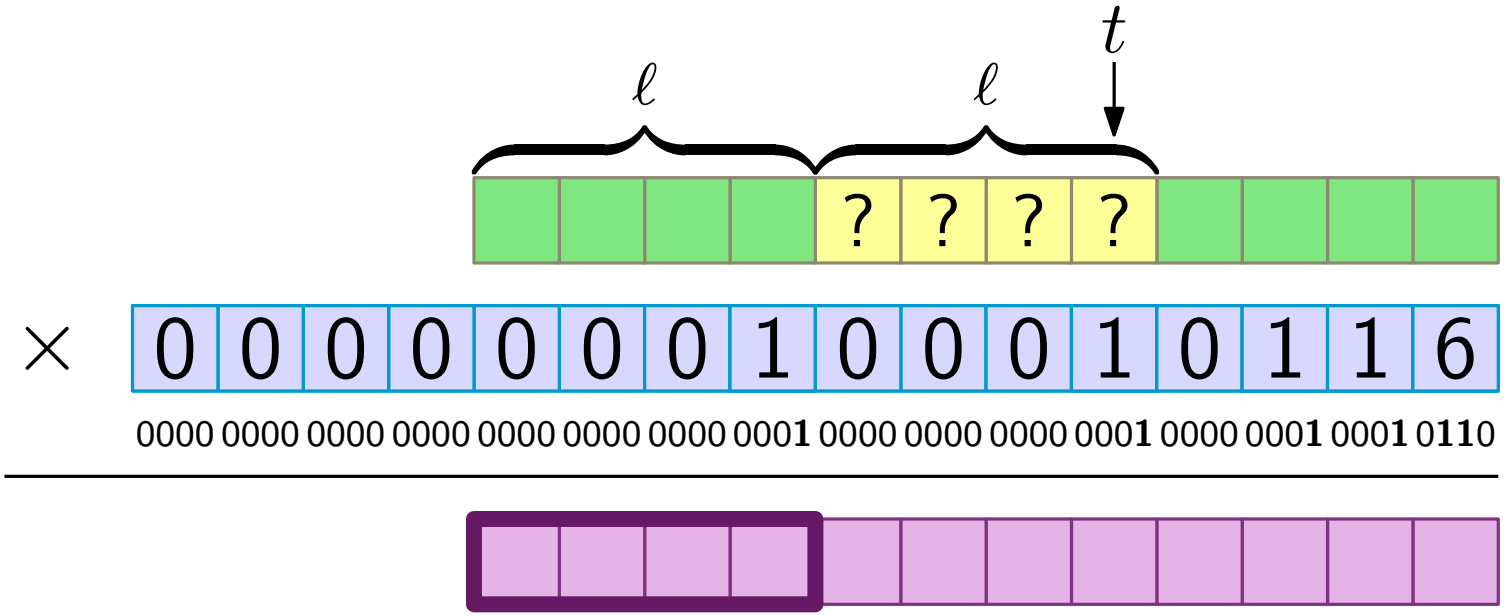


Arriving digit (fixed value)

? Unknown digit chosen uniformly at random from $[q]$

Output digit

Multiplication

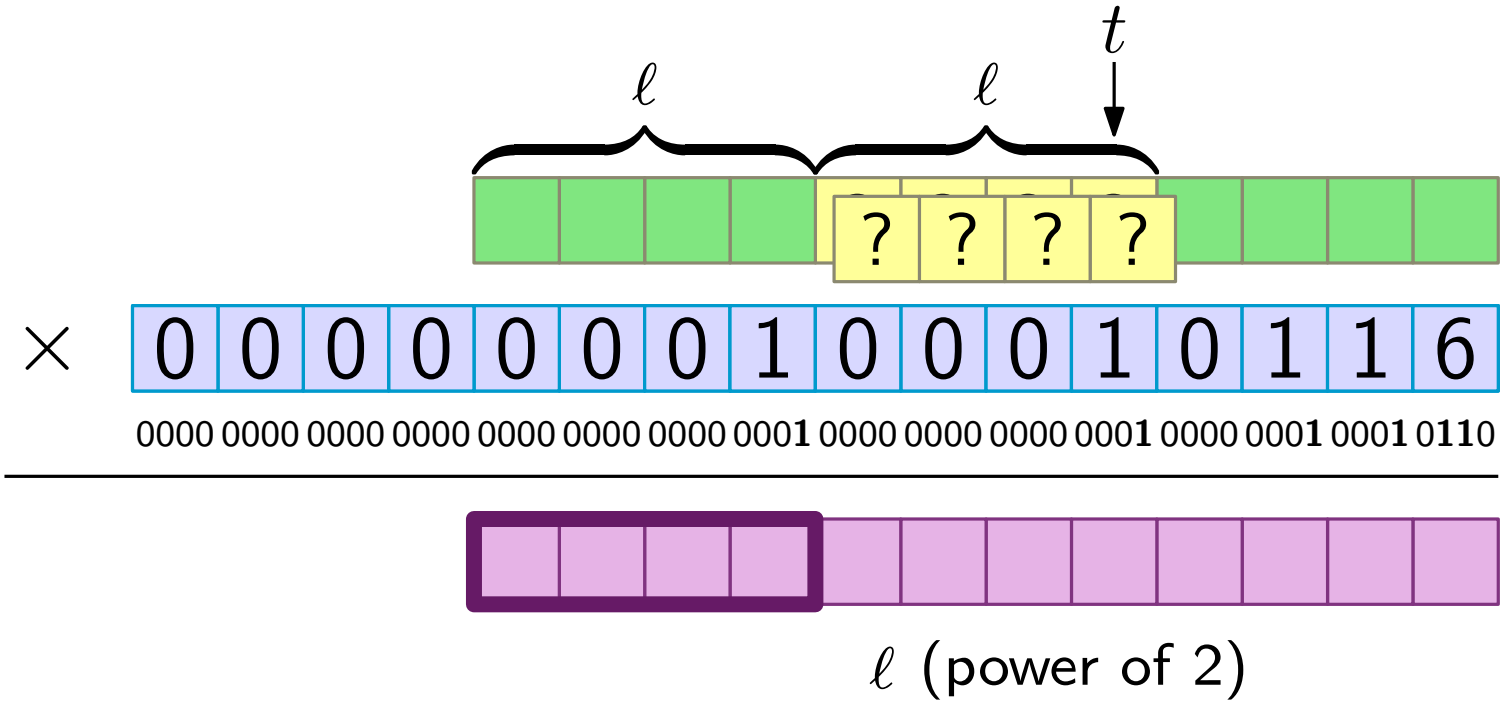


Arriving digit (fixed value)

? Unknown digit chosen uniformly at random from $[q]$

Output digit

Multiplication

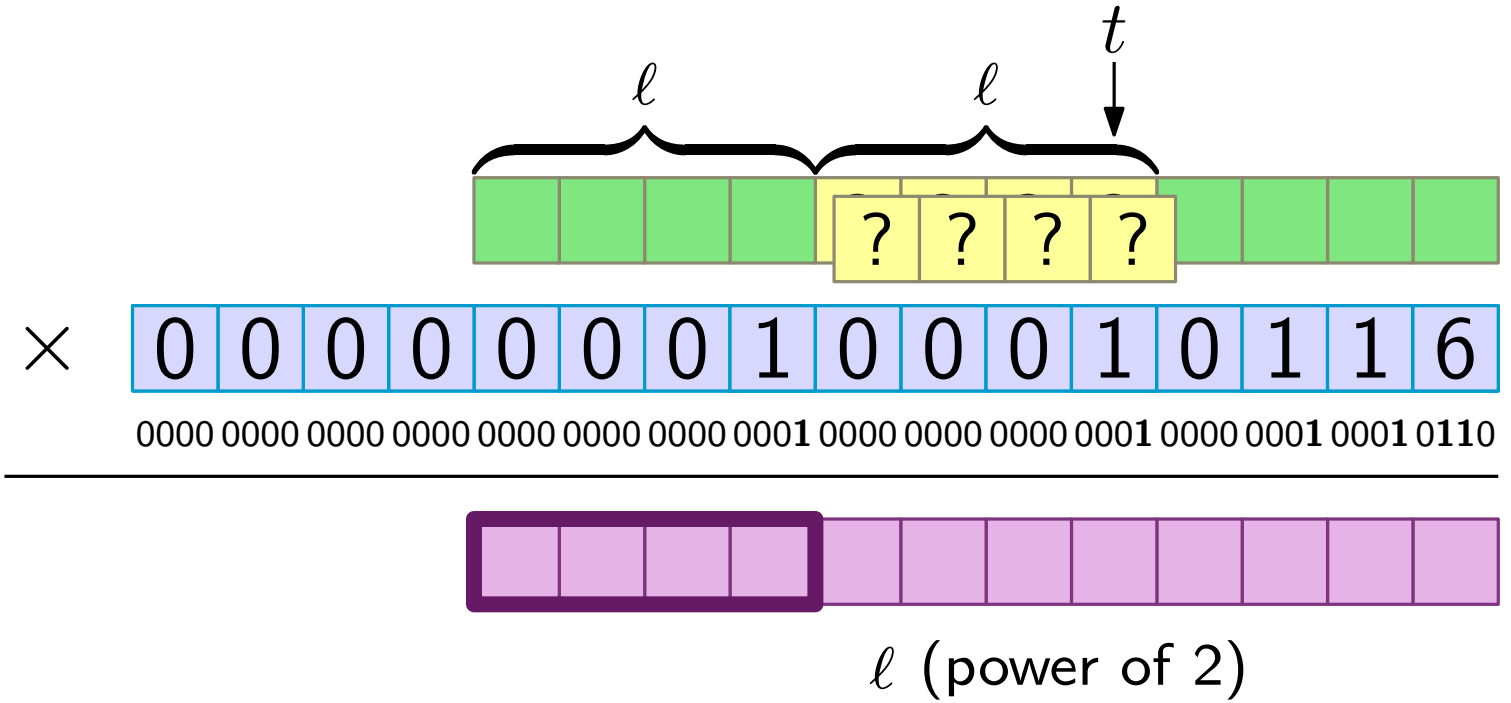


ℓ (power of 2)

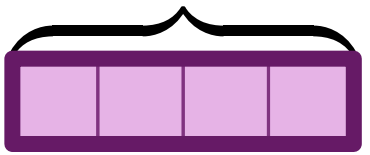
Under any fixed digits and any ,

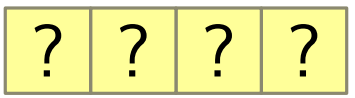
there can be at most **two** possible values of ? ? ? ?

Multiplication

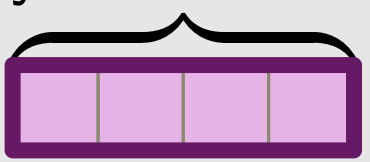


ℓ (power of 2)

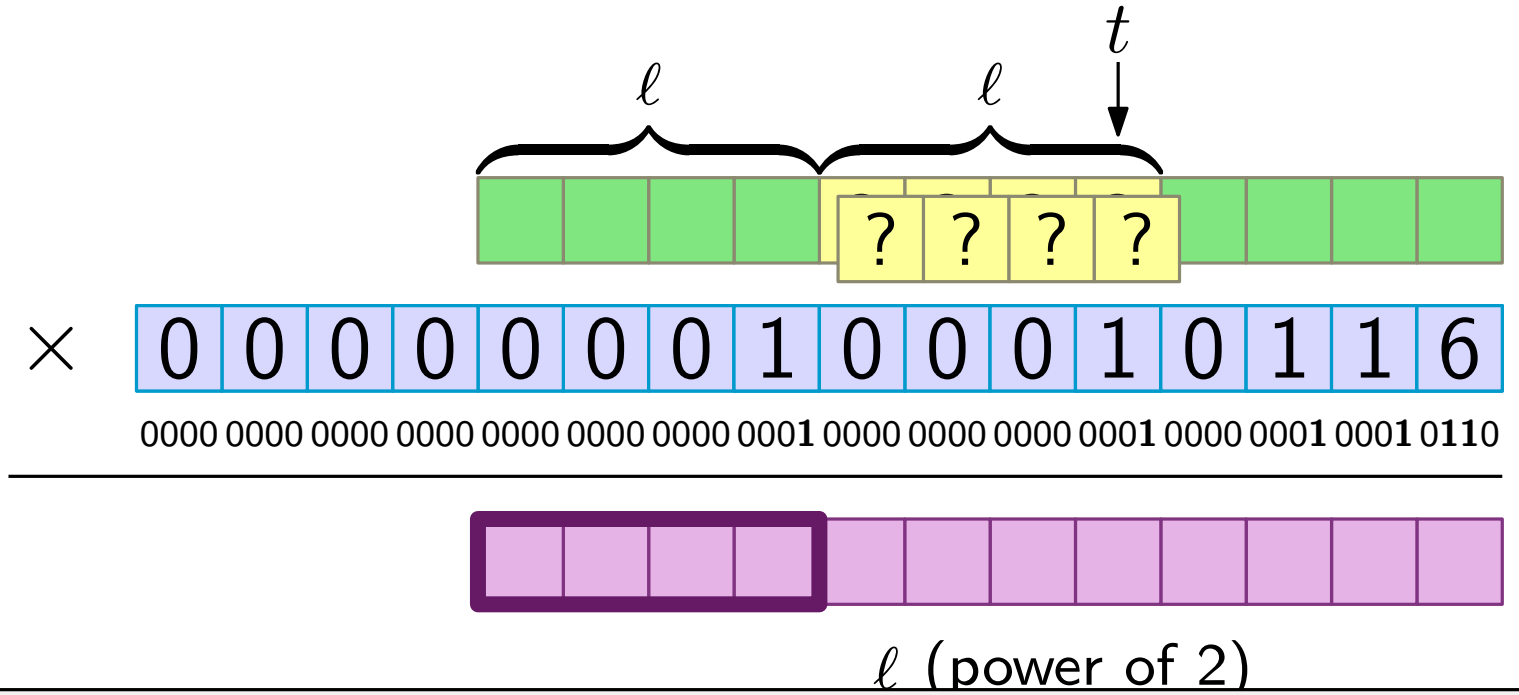
Under any fixed digits  and any  ,

there can be at most **two** possible values of 

The conditional entropy

$H($the outputs during  | all  fixed$) \geqslant \frac{\ell}{2}\delta - \frac{1}{2}$

Multiplication

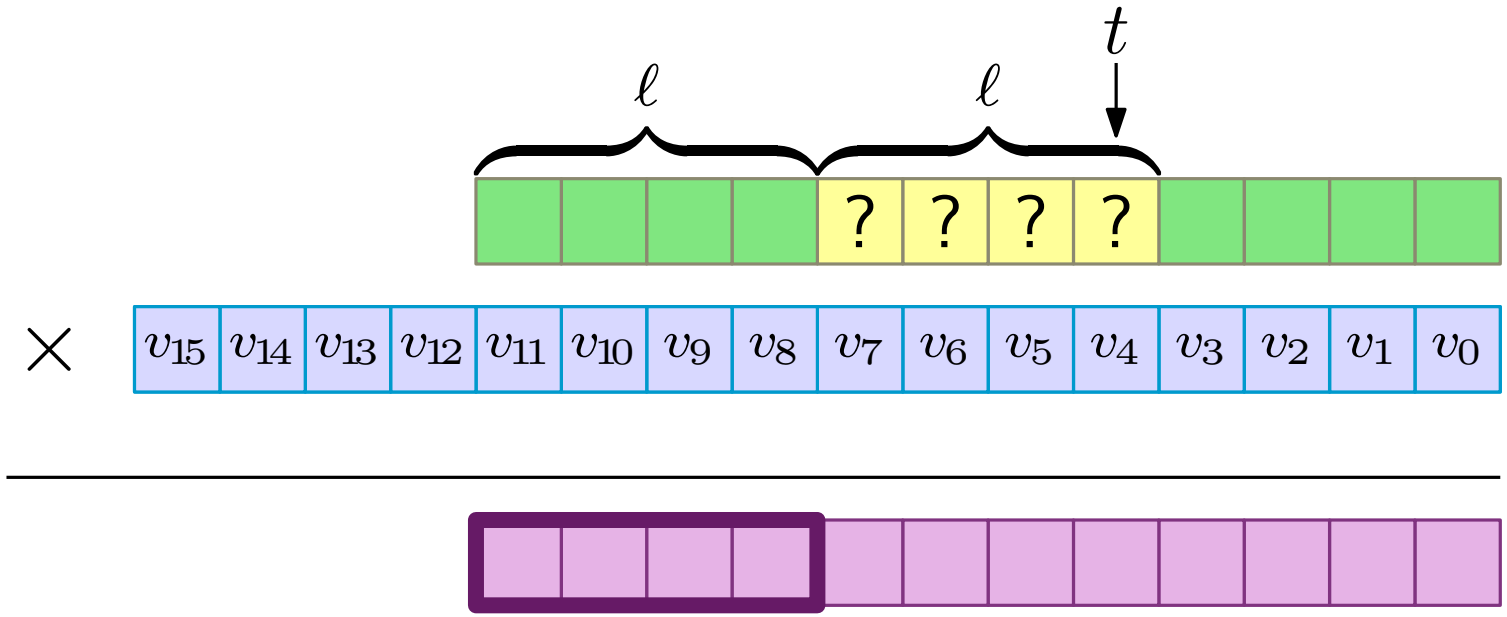


Using the information transfer method, we get the time lower bound: $\Omega\left(\frac{\delta}{w} \cdot n \log n\right)$

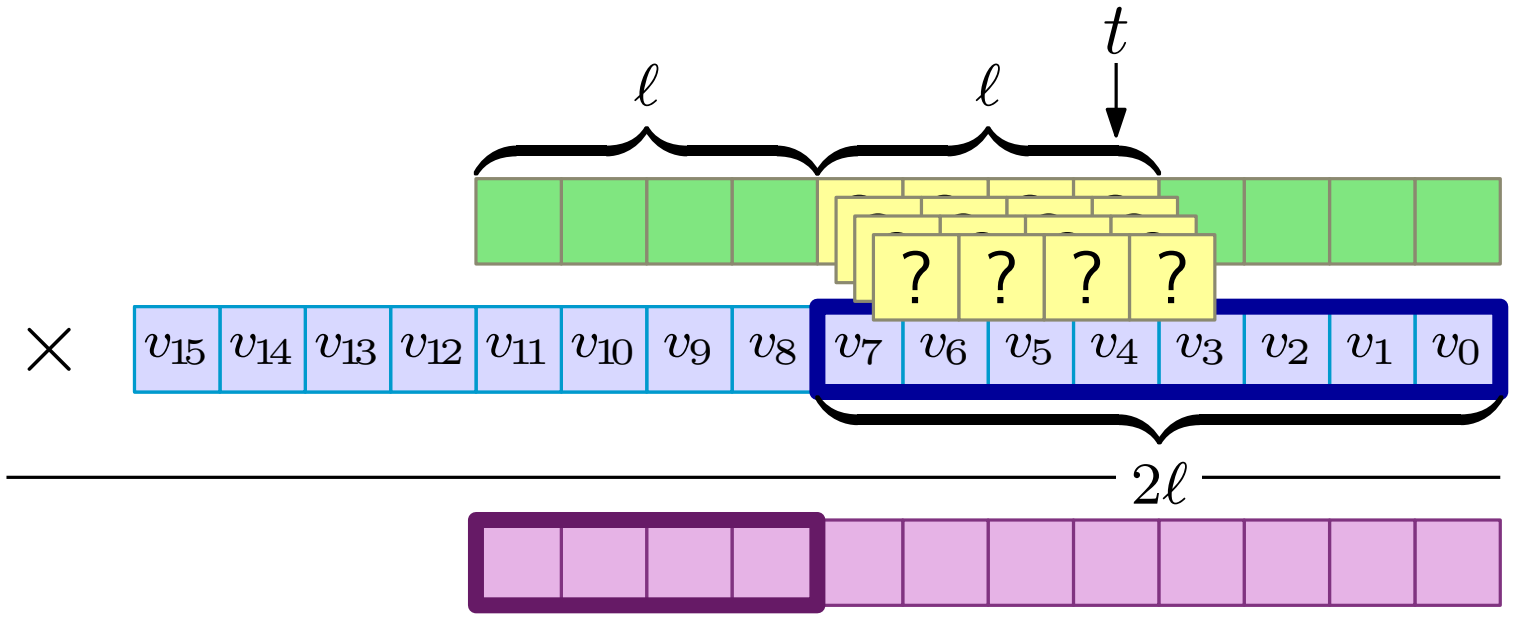
The conditional entropy

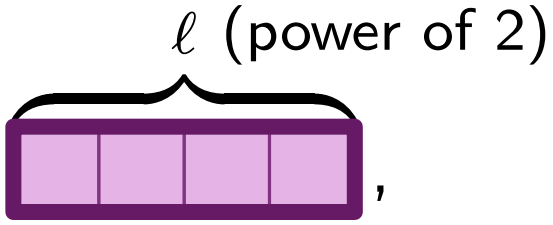
$H($the outputs during $\overbrace{\quad}^{\ell}$ | all ■ fixed$) \geqslant \frac{\ell}{2}\delta - \frac{1}{2}$

Multiplication

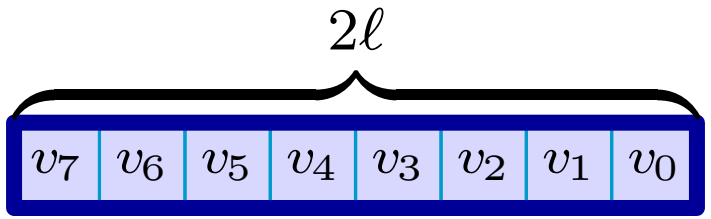


Multiplication

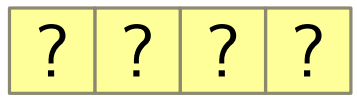




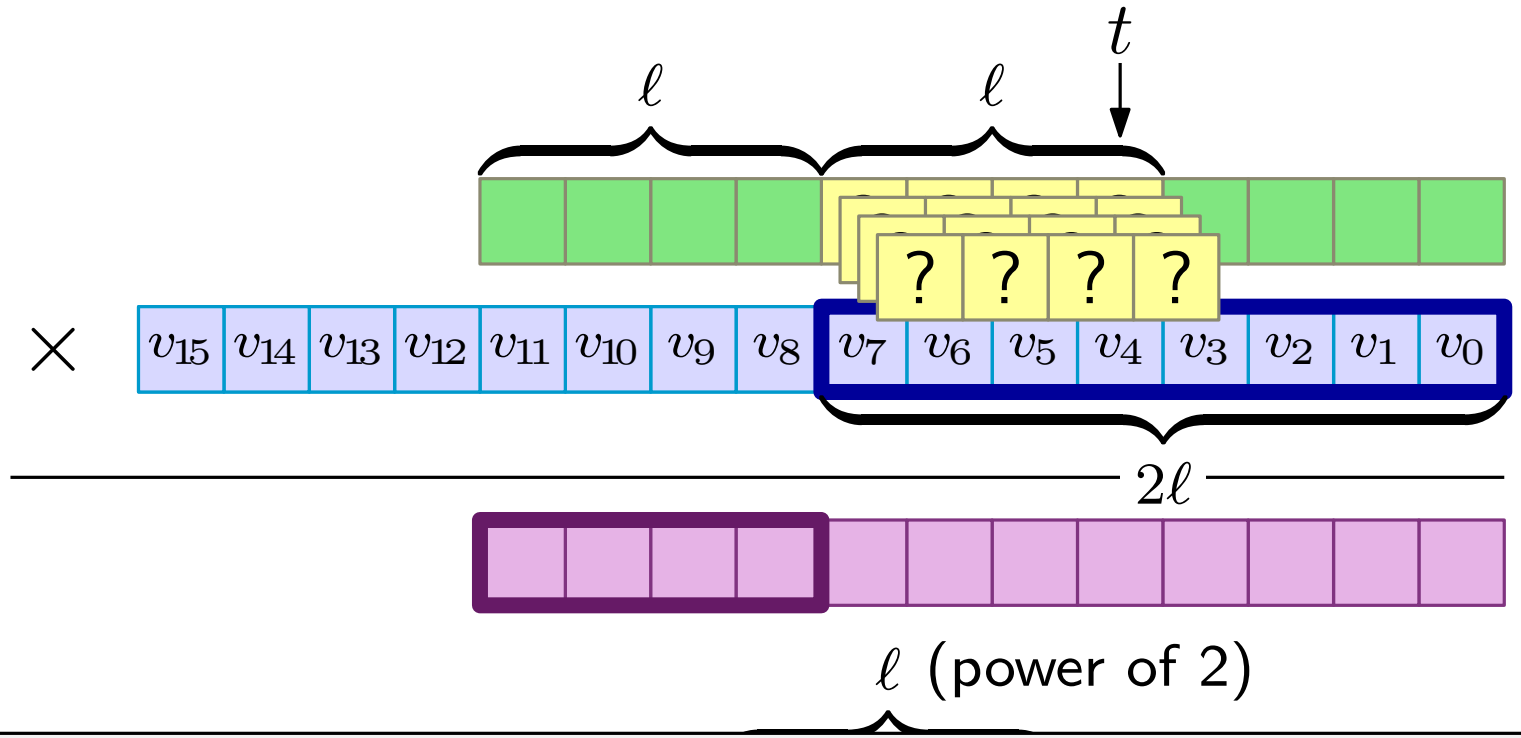
Under any fixed digits  and any ,



for at least half of all possible values

there can be at most **four** possible values of 

Multiplication



For a random fixed number, the time lower bound is also $\Omega\left(\frac{\delta}{w} \cdot n \log n\right)$

there can be at most **four** possible values of | ? | ? | ? | ? |

Thank you!