

Quantum search of partially ordered sets

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Introduction

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- Given an arbitrary non-zero function $f : [n] \mapsto \{0, 1\}$, Grover's quantum algorithm finds an x such that $f(x) = 1$, with constant probability, using only $O(\sqrt{n})$ queries to f .
- Important extension to **amplitude amplification**: given a probabilistic algorithm A that succeeds with probability p , and the ability to recognise a correct solution, can output a correct solution with probability $\Omega(1)$ using only $O(1/\sqrt{p})$ uses of A [Brassard et al '00].

More complicated search

Grover's algorithm is already a useful primitive to speed up more complicated classical algorithms. For example, we can:

- Find the minimum element in a set of n integers in $O(\sqrt{n})$ time [Dürr, Høyer '96],
- Find a collision in a $2 \rightarrow 1$ function $f : [2n] \mapsto [n]$ in $O(n^{1/3})$ time [Brassard et al '97],
- Find a spanning tree in an n -vertex graph in $O(n^{3/2})$ time (adjacency matrix model) [Dürr et al '04],
- ...

All these applications work by finding a part of the problem in question that's essentially **unstructured**, and running Grover search on this.

Recursive quantum search?

- Recursion is a fundamental tool in classical algorithms, but is little used in quantum algorithms.
- Could we speed up a search problem that has some kind of recursive structure?
- (Caution: already known that quantum search of an ordered n -element list requires $\Omega(\log n)$ time [Høyer et al '01])

Classical recursive search

Consider the problem of (classical) search of an abstract “database” D_n , parametrised by a problem size n , for a “marked” element. Let D_n have the following characteristics:

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- If there is a marked element in the original database, then there is a marked element in exactly one of these sub-databases.
- Each division into sub-databases uses time $f(n)$, where $f(n) = O(n^{1-\epsilon})$ for some $\epsilon > 0$.

What is the time $T(n)$ to find an element in D_n ?

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We would like to find a **quantum version** of this recurrence. Can we get a speed-up by searching the sub-databases in quantum parallel?

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- If there is a marked element in the original database, then there is a marked element in exactly one of these sub-databases.
- Each division into sub-databases uses time $f(n)$, where $f(n) = O(n^{1/2-\epsilon})$ for some $\epsilon > 0$.

Then there is a quantum algorithm that finds an element in D_n with constant probability in time $T(n) = O(\sqrt{n})$.

Finding the intersection of two sorted lists

Problem

Given monotone functions $f : [n] \mapsto \mathbb{Z}$, $g : [n] \mapsto \mathbb{Z}$, output a y such that $f(x) = g(x') = y$ for some x, x' , if such a y exists.

1	2	4	4	8	9	10
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3	4	5	6	6	7	8
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- Obvious classical lower bound is $2n$ queries (have to read all the input in)
- “Obvious” quantum algorithm uses $O(\sqrt{n} \log n)$ queries
- [Buhrman et al '05] gave an ingenious $O(\sqrt{nc}^{\log^* n})$ algorithm
- Lower bound is $\Omega(\sqrt{n})$ queries

We give an algorithm matching this lower bound.

A recursive classical algorithm

Idea: reduce the problem to searching in a 2d array sorted along rows and columns.

- Consider a notional $n \times n$ array T where $T(x, y) = f(x) - g(n + 1 - y)$.
- Then finding a zero in T finds a match in the two lists.

	8	7	6	6	5	4	3
1	-7	-6	-5	-5	-4	-3	-2
2	-6	-5	-4	-4	-3	-2	-1
4	-4	-3	-2	-2	-1	0	1
4	-4	-3	-2	-2	-1	0	1
8	0	1	2	2	3	4	5
9	1	2	3	3	4	5	6
10	2	3	4	4	5	6	7

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- 1 Perform binary search on the middle row/column of A .
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Can show $T(n) \leq O(\log n) + 2T(n/2) = O(n)$.

(a different optimal classical algorithm was already known [Linial, Saks '85], but seems harder to “make quantum”)

Example

1	3	5	10	13
2	4	7	11	14
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12	16	17	20	24
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(**green**: integer to search for, **yellow**: not searched yet, **blue**: currently being searched, **red**: discarded)

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- Each division into sub-databases uses time $f(n)$, where $f(n) = O(n^{1/2-\epsilon})$ for some $\epsilon > 0$. ✓
- If there is a marked element in the original database, then there is a marked element in exactly one of these sub-databases. ✗

We might have more than one zero in the array.

Finishing off the algorithm

Problem: The recursive quantum search algorithm can only cope with at most one marked element.

Solution:

- Note that the zeroes only occur in rectangular blocks, with at most one block per row and column
- If there's only one such "zero block", can modify the search algorithm to pretend that the block contains one element
- If not, to reduce to the single-block case, repeatedly throw away random rows and columns over several rounds
- Can show that with constant probability, one round will have only one zero block remaining
- Can also show that the asymptotic query complexity isn't hurt by doing this.

Proof of the recursive quantum search theorem (sketch)

We use a iterations of amplitude amplification on the following probabilistic classical algorithm.

- 1 Perform recursive search down to l levels, i.e. split the database l times into k^l pieces.
- 2 Pick a piece at random.
- 3 Call yourself on that piece.
- 4 Repeat until the marked element is found.

Then perform a number of iterations of amplitude amplification on the whole algorithm such that the success probability is $\Omega(1)$.

The parameters a and l are picked to ensure that the whole algorithm takes $O(\sqrt{n})$ time.

Proof of the recursive quantum search theorem (sketch)

Why this somewhat complicated process?

- What happens if we just perform l levels of recursion then use amplitude amplification on the resulting k^l sub-databases?
- We'd get $T(n) \leq k^l O(n^{1/2-\epsilon}) + O(k^{l/2}) T(n/k^l) = O(n^{1/2+c})$
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 - for some constant c that turns up because of the hidden constant in the big- O notation
- But it's more efficient to do fewer iterations of amplitude amplification.
- So we only do "a small amount" of amplification at each recursive step and then do "lots" at the end.
- Based on a powerful result of [\[Aaronson and Ambainis '05\]](#) on quantum search of spatial regions.

Extensions

- The quantum algorithm for finding an integer in a $n \times n$ array of distinct integers immediately extends to a d -dimensional $n \times n \times \cdots \times n$ array sorted in each dimension (complexity is $O(n^{(d-1)/2})$)
- This is a special case of a more general problem: quantum search of **partially ordered sets** (posets).
- One can show general upper and lower bounds for this task (summary: quantum computers can achieve at most a quadratic speed-up (approx) for **any** poset, and barely any speed-up at all for some posets).

Summary and further work

- We have outlined a general approach for achieving a quantum speed-up from recursive classical search algorithms.
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Future work?

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- Further applications? Finding problems where the speed-up is more dramatic?

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Further reading: “Quantum search of partially ordered sets”,
<http://arxiv.org/abs/quant-ph/0702196>

Bristol Summer School on Probabilistic Techniques in Computer Science

6-11 July 2008



- Keynote speaker: Bela Bollobás.
- Topics include: randomised algorithms, communication complexity, concentration of measure, data stream algorithms, ...

<http://www.cs.bris.ac.uk/probtcs08/>