

The Expressive Power of Binary Submodular Functions

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Problem



Which submodular polynomials can be expressed by (or decomposed into) quadratic submodular polynomials?

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Degree-4 polynomials.

Motivation



Problem considered in:

- ▶ artificial intelligence
- ▶ computer vision
- ▶ optimisation

Quadratic submodular polynomials

=

minimisation via (s, t) -Min-Cut.

Minimisation of submodular polynomials



▶ PTIME

[ellipsoid method, GLS'81]

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- ▶ 0/1-valued [Creignou, Khanna, Sudan '01]

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- ▶ 0/1-valued [Creignou, Khanna, Sudan '01]
- ▶ other non-Boolean results

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Which submodular **polynomials** can be expressed by quadratic submodular polynomials?

pseudo-Boolean: $\{0, 1\}$, coefficients in \mathbb{R}

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- ▶ key concept in combinatorial optimisation

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- ▶ DIGRAPH MIN-COST-HOM: min-max ordering
- ▶ VCSP, MAX-CSP: tractability

Submodularity



$$p(x_1, \dots, x_n) = a_0 + \sum_{i=1}^n a_i x_i + \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j$$

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Recognition easy.

Submodularity, cont'd



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Submodularity, cont'd



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$$\delta_1(\mathbf{x}) = p(1, \mathbf{x}) - p(0, \mathbf{x})$$

Submodularity, cont'd



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p is **submodular** $\Leftrightarrow \forall i, j : \delta_{ij}(\mathbf{x}) \leq 0$

$$\delta_{1,2}(\mathbf{x}) = p(1, 1, \mathbf{x}) - p(1, 0, \mathbf{x}) - p(0, 1, \mathbf{x}) + p(0, 0, \mathbf{x})$$

Submodularity, cont'd



$$p(x_1, \dots, x_n) = a_0 + \sum_{i=1}^n a_i x_i + \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j + \sum_{1 \leq i < j < k \leq n} a_{ijk} x_i x_j x_k + \dots + \sum_{|I|=k} a_I x_I$$

p is **submodular** $\Leftrightarrow \forall i, j : \delta_{ij}(\mathbf{x}) \leq 0$

$$p(1, 1, \mathbf{x}) + p(0, 0, \mathbf{x}) \leq p(0, 1, \mathbf{x}) + p(1, 0, \mathbf{x})$$

Submodularity, cont'd



$$p(x_1, \dots, x_n) = a_0 + \sum_{i=1}^n a_i x_i + \sum_{1 \leq i < j \leq n} a_{ij} x_i x_j + \\ \sum_{1 \leq i < j < k \leq n} a_{ijk} x_i x_j x_k + \dots + \sum_{|I|=k} a_I x_I$$

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Recognition hard even for $k=4$.

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Expressibility



$$p \in \langle L \rangle \Leftrightarrow p(\mathbf{x}) = \min_{\mathbf{z}} \sum_i p_i(\mathbf{x}, \mathbf{z}) + c \quad (p_i \in L, c \in \mathbb{R})$$

Example 1



Let

$$p(x_1, \dots, x_{100}) = -x_2x_5x_6x_9.$$

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Then,

$$p(x_1, \dots, x_{100}) = \min_{y \in \{0,1\}} y(3 - x_2 - x_5 - x_6 - x_9).$$

Example 1, more generally



Let $I \subseteq [n]$, $|I| = k$, and

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Then,

$$p(x_1, \dots, x_n) = \min_{y \in \{0,1\}} y(k - 1 - \sum_{i \in I} x_i).$$

Example 2



$$p(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 - x_1x_2 - x_1x_3x_4 - x_2x_3x_4$$

Example 2



$$p(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 - x_1x_2 - x_1x_3x_4 - x_2x_3x_4$$

$$p(x_1, x_2, x_3, x_4) = \min_{y_1, y_2} (y_1 + 2y_2 - y_1y_2 - y_1x_1 - y_1x_2 - y_2x_3 - y_2x_4)$$

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2 extra variables, 1 is provably not enough

Example 3



$$\begin{aligned} p(x_1, x_2, x_3, x_4) = & -x_1x_2x_3x_4 + x_1x_3x_4 + x_2x_3x_4 \\ & -x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_3x_4 \end{aligned}$$

Example 3



$$p(x_1, x_2, x_3, x_4) = -x_1x_2x_3x_4 + x_1x_3x_4 + x_2x_3x_4 \\ - x_1x_3 - x_1x_4 - x_2x_3 - x_2x_4 - x_3x_4$$

Not expressible with any number of extra variables!

Theorem [Cooper, Cohen, Jeavons '06]

1. $p \in \langle L \rangle \Leftrightarrow \text{fPol}(L) \subseteq \text{fPol}(\{p\})$
2. $p \in \langle L \rangle \Rightarrow p \in \langle L \rangle$ with at most 2^{2^k} extra variables

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Mul = Multimorphisms: $\{0, 1\}^k \rightarrow \{0, 1\}^k$

Notation



Sub_k = submodular polynomials of degree k

Sub = all submodular polynomials

Original plan



$$\text{fPol}(\text{Sub}) = \text{fPol}(\text{Sub}_2)$$

Original plan



$$\begin{aligned} \text{fPol}(\text{Sub}) &= \text{fPol}(\text{Sub}_2) \\ &\Downarrow \\ \text{Sub} &= \langle \text{Sub}_2 \rangle \end{aligned}$$

Original plan



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\Downarrow

$$\text{Sub} = \langle \text{Sub}_2 \rangle$$

\Downarrow

World is nice.

Original plan



$\text{fPol}(\text{Sub}) \neq \text{fPol}(\text{Sub}_2)$

\Downarrow

$\text{Sub} = \langle \text{Sub}_2 \rangle$

\Downarrow

World is **still** nice.

Theorem 1 [Ž., Cohen, Jeavons '08]

Let $p \in \text{Sub}_4$. Then the following are equivalent:

1. $p \in \langle \text{Sub}_2 \rangle$
2. $\forall \{i, j\}, \{k, l\} \subseteq [n]$ distinct : $a_{ij} + a_{kl} + a_{ijk} + a_{ijl} \leq 0$

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3. $p \in \text{Cone}(\text{Fans}_4)$
4. $\mathcal{F}_{sep} \in \text{Mul}(\{p\})$

Proof



- ▶ $(3) \Rightarrow (1) : p \in \text{Cone}(\text{Fans}_4) \Rightarrow p \in \langle \text{Sub}_2 \rangle$

Theorem 2: All Fans are expressible.

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Theorem 3: Characterisation of $\text{Mul}(\text{Sub}_2)$.

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- ▶ $(4) \Rightarrow (3) : \mathcal{F}_{sep} \in \text{Mul}(\{p\}) \Rightarrow p \in \text{Cone}(\text{Fans}_4)$
The same polyhedra.

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The same polyhedra.
- ▶ (4) \Leftrightarrow (2) : $\mathcal{F}_{sep} \in \text{Mul}(\{p\}) \Leftrightarrow a_{ij} + a_{kl} + a_{ijk} + a_{ijl} \leq 4$
Translation of different representations.

Non-expressibility over Sub_4



Essentially one reason:

Non-expressibility over Sub_4



Essentially one reason:

$$qin_{12} = \begin{cases} -1 & 0000, 1111 \\ +1 & 1100 \\ 0 & o/w \end{cases}$$

Expressibility of Fans



Theorem 2 [Ž., Cohen, Jeavons '08]
Fans of all arities are expressible over Sub_2 .

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Optimal number of variables.

Characterisation of $\text{Mul}(\text{Sub}_2)$



Theorem 3 [Ž., Cohen, Jeavons '08]

$\mathcal{F} \in \text{Mul}(\text{Sub}_2) \Leftrightarrow$

\mathcal{F} is conservative Hamming distance non-increasing.

Characterisation of $\text{Mul}(\text{Sub}_2)$



Theorem 3 [Ž., Cohen, Jeavons '08]

$\mathcal{F} \in \text{Mul}(\text{Sub}_2) \Leftrightarrow$

\mathcal{F} is conservative Hamming distance non-increasing.

Helped to identify \mathcal{F}_{sep} .

Recognition



Recognition $p \in \langle \text{Sub}_2 \rangle$ easy assuming $p \in \text{Sub}_4$.

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Recognition Sub_4 co-NP-C. [Gallo, Simeone; Crama '88]

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Theorem 4 [Ž., Cohen, Jeavons '08]

1. Testing $p \in \text{Sub}_4 \cap \langle \text{Sub}_2 \rangle$ is co-NP-C.

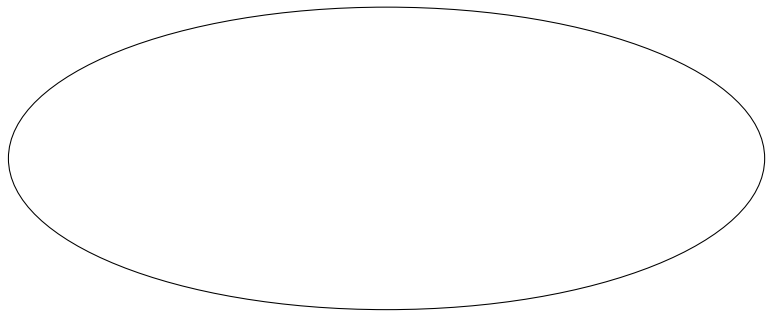
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Recognition Sub_4 co-NP-C. [Gallo, Simeone; Crama '88]

Theorem 4 [Ž., Cohen, Jeavons '08]

1. Testing $p \in \text{Sub}_4 \cap \langle \text{Sub}_2 \rangle$ is co-NP-C.
2. Not all extreme rays are Fans (Promislow & Young).

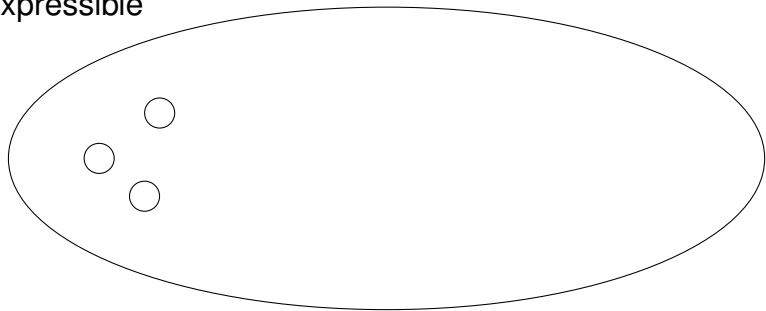
Submodular polynomials



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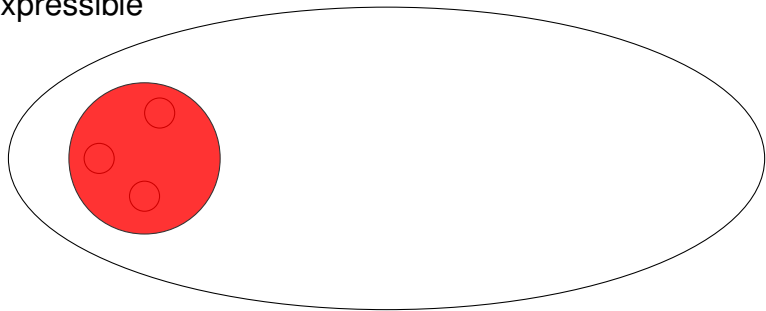
Expressible



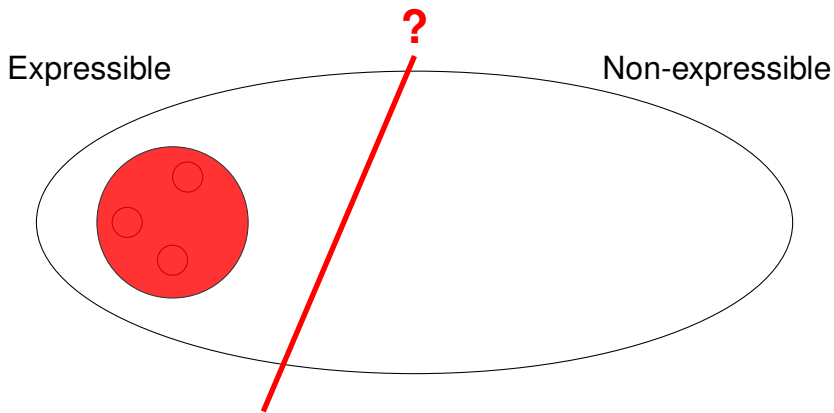
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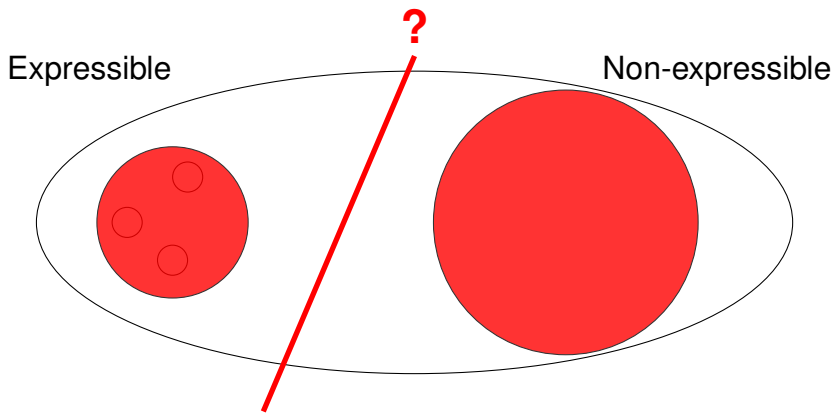
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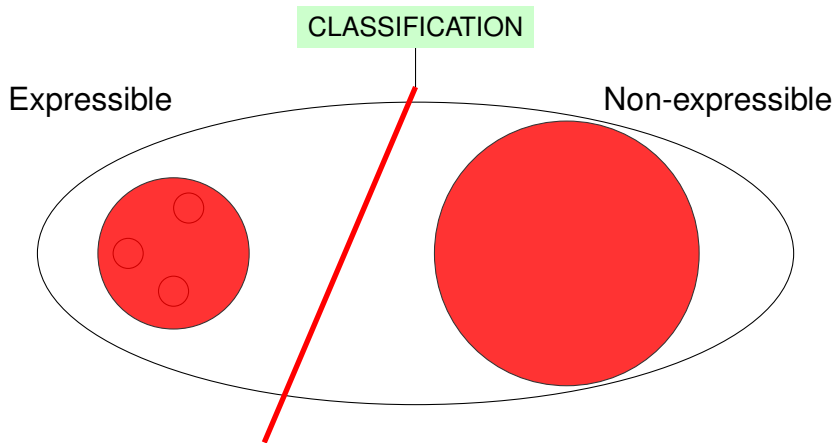
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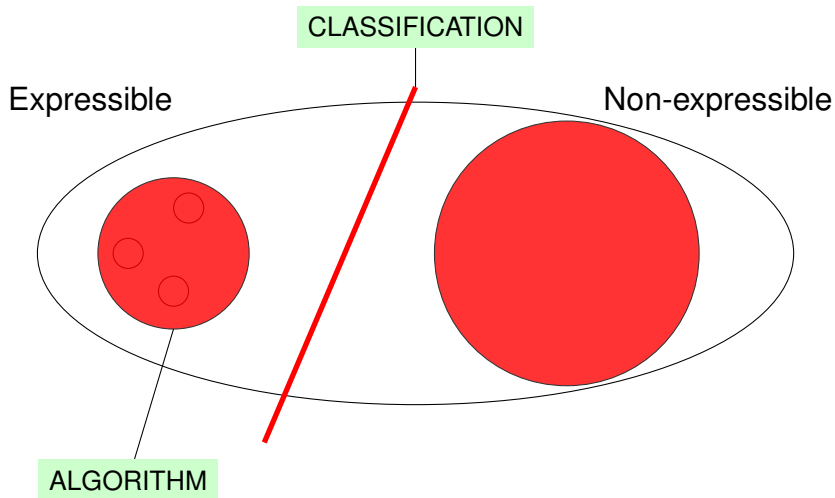
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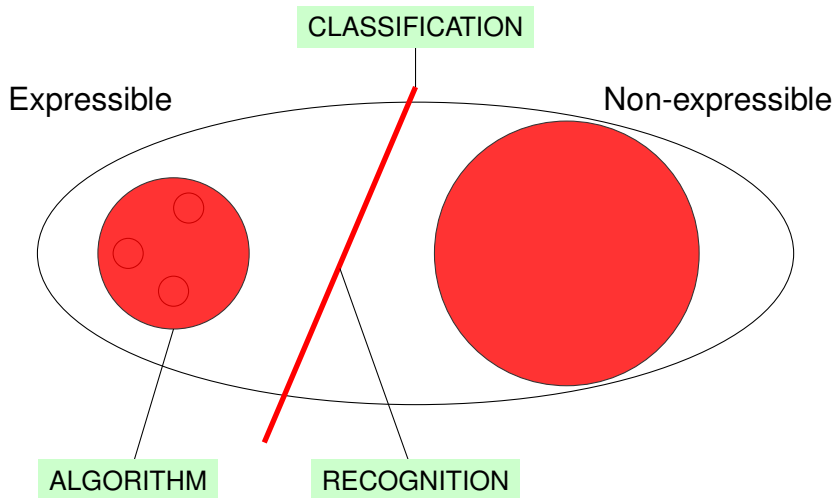
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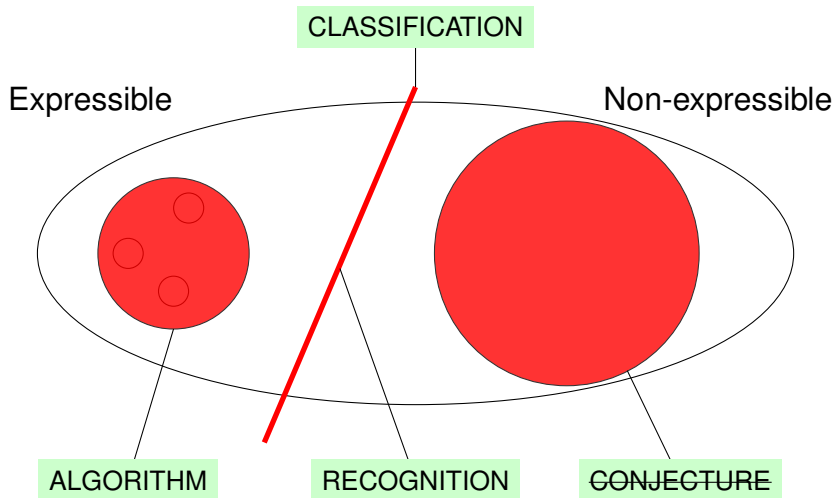
Submodular polynomials



Submodular polynomials



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Multimorphism definition



$\mathcal{F} = \langle f_1, \dots, f_k \rangle : D^k \rightarrow D^k$ is a multimorphism of p in m variables if and only if for all $t_1, \dots, t_k \in D^m$,

$$\sum_{i=1}^k p(t_i) \geq \sum_{i=1}^k p(f_i(t_1, \dots, t_k)).$$

(where f_i is applied coordinatewise)

